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PROPAGATION OF LONG WAVELENGTH DISTURBANCES IN A PLASMA

Richard L. Liboff

October 2, 1961

AEC Research and Development Report

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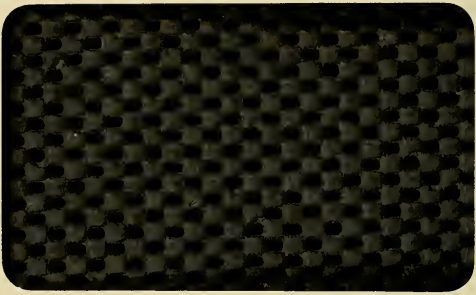

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Abstract

The long wavelength modes of excitation of a two component plasma in a steady magnetic field are examined. Two linearized Boltzmann equations are given with collision terms whoch are coupled through the difference in temperatures and difference in velocities of the two gases. A formal means of classification of phenomena is described in terms of the nature of the roots about $k = 0$. Two types of behavior are uncovered: MHD, Magnetohydrodynamics, which includes finite phase speed phenomena near $k = 0$; and PEM, Plasma Electromagnetics, which includes infinite phase speed phenomena near $k = 0$. The dissipative effects of collisions are included. In the limit of vanishing collision frequency, roots previously obtained are recaptured. The relevance of the pertinent domains are discussed and the complex interplay between the "fast" and "Alfven" modes of MHD and the plasma-magnetic modes of PEM are demonstrated. Equations macroscopic in appearance are derived which include the effects of the initial configuration. In the limit of large collision frequency these equations reduce to standard forms. The dispersion of the N^{th} order Larmor resonance is given which includes the effect of the mass ratio. A discussion of the transfer equations of a plasma is included.

Acknowledgements

The author is most indebted to Professor Harold Grad for suggesting this investigation, and for being a constant guide and source of encouragement throughout the writing of this paper. To Professor Bruno Zumino he is grateful for many illuminating discussions on these and related topics. Dr. Lawrence Sirovich was instrumental in the initial formulation of the collision model. Dr. Harold Weitzner was equally instrumental in the closing sections, specifically with regard to the proper physical interpretation of the results. The author is especially grateful to Dr. Clifford S. Gardner, whose constant ability to offer the clear and "obvious" analysis contributed so greatly to this endeavor.

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PROPAGATION OF LONG WAVELENGTH DISTURBANCES IN A PLASMA

Richard L. Liboff

Errata

1) p. 25 eq. 23: $n\kappa T$ should read $\overline{n}\kappa\overline{T}$.

2) p. 44 eq. 14: exponential should read

$$\exp\left\{\bar{a}\alpha' - \frac{ik_\perp \xi_\perp}{\Omega}[\sin\alpha - \sin(\alpha + \alpha']\right\} .$$

3) p. 45 eq. 20: $2^{4\pi\bar{a}}$ should read $e^{4\pi\bar{a}}$.

4) p. 58 line 9: $u_\perp$ should read $\hat{u}_\perp$.

5) p. 70 line 3: $3a^2 + 1$ should read $3a^2 - 1$.

6) p. 73 add $-2ik_\perp \times B[S^2]^{(0)}(\nu n + \mathcal{L})$ to eq. 34 .

7) p. 81 eq. 7: $\mu_o J = \dot{E}/c^2$ should read $\mu_o J + \dot{E}/c^2$.

8) p. 91 eq. 28: $1/1\omega$ should read $1/i\omega$.

9) p. 110 footnote page reference is p. 38
line 16: change p^2 to $\wp^2$.

10) p. 130 In the equation for I_ℓ change $2in \sin\psi$ to $2iy \sin\psi$.

11) p. 5 General Notation: $X = \hat{X} - X$ should read $\Delta X = \hat{X} - X$.

12) p. 88 eq. 19: first term should read $-i\omega\tilde{k}\rho_o U^{(o)}$.

13) p. 131 omit third equation.

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PROPAGATION OF LONG WAVELENGTH DISTURBANCES IN A PLASMA

R. L. Liboff

Errata

1) p. 5 General Notation: $X = \hat{X}-X$ should read $\Delta X = \hat{X}-X$.

2) p. 14 Diagram: Lower left corner, change $\omega^2 = 0$ to $(PO)^{(2)}$.

3) p. 17 Line 13: change to: "..., then these roots collapse to the four plasma oscillations $\omega_{\pm}^2 = \omega_o^2$."

4) Sec. B-I eq. 10: change the first n to $\bar{n}$.
eq. 19: change W to $\bar{W}$.
eq. 23: change $n\kappa T$ to $\bar{n}\kappa\bar{T}$.

5) Sec.B-II eq. 2: change $H^{(2)}$ to $\hat{H}^{(2)}$.
eq. 10: The right hand side includes the factor $\mathfrak{M}$.
eq. 23: change Ω to $\bar{\Omega}$.

6) Sec.C-I eq. 3: In equation for $\hat{h}_1$, change u to $\hat{u}$.
eq. 14: A minus sign should follow $\bar{a}\alpha'$.
eq. 20: change $2^{4\pi\bar{a}}$ to $e^{4\pi\bar{a}}$.
eq. 32 and 34: change ϕ_o^2 to ϕ_o .

7) Sec.C-II eq. 1: change h to $\hbar$.
eq. 10: change $\xi\cdot B\,\hbar_1$ to $\xi\cdot B\times\hbar_1$.
eq. 18: change $\frac{1}{2}$ factor to 1.
p. 58, line 9: change $u_\perp = \hat{F}\,\Delta u$ to $\hat{u}_\perp = \hat{F}\,\Delta u$.
eq. 30: should read: $\frac{1}{\tau} = \frac{i}{N-1} + \frac{1}{\hat{a}-i} + \frac{1}{\bar{v}_3}$.

8) Sec.C-III eq. 8: change $\cos^N$ to $\cos^M$.
eqs. 13: should include $\xi_{\parallel}$ factor in integrands.
eq. 25: First k^2 should be $k_{\parallel}^2$ and the second k^2 should be $k_{\perp}^2$.
eq. 27.1: $\frac{\partial}{\partial T}$ should read $\frac{\partial}{\partial t}$.
eq. 30: change C^2 to $\hat{C}^2$.
eq. 34: add the term $-2ik_{\perp} \times B[S^2]^{(0)}(vn + \mathcal{T})$.
eq. 39: Insert a plus sign between $(vn + \mathcal{T})$ and h

9) Sec.C-IV eq. 7: change second equals sign to a plus sign.
eq. 19: first term should read $-i\omega\tilde{k}\rho_0 U^{(0)}$
eq. 28: change 1 to i.
eq. 30: change $J \times B$ to $J \times \tilde{B}$.
eq. 38: change Ω to $\tilde{\Omega}$.
eqs. 40: Left sides should include the factor 2 and the radicals should be multiplied by i
p. 103: line seven, omit k^2A^2.
eq. 64.1: all speeds should be squared.
eq. 69: change Ω to $\tilde{\Omega}^2$.

Appendix C. p. 130: In the equation for I_ℓ change 2in to 2iy

Appendix D. p. 131: Omit third equation.

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List of Variables

General Notation:

If X is a variable of the system, then,

X^* = the transform of X,

$\hat{X}$ = the related ion variable,

$X = \hat{X} - X$ = the difference between ion and electron values.

$\overline{X}$ = the total X, i.e., the equilibrium value plus the perturbation,

X_o = the equilibrium value of X.

Symbol	Description	Page
$\overline{a}$	Integration variable	43
a	$\overline{a}(k = 0)$	50
a_o^2	Sound speed (squared)	89
A, A'	Alfven speeds	89
A_{ij}	System matrix	46
B, B_o	Magnetic field (perturbation, unit vector; equilibrium)	21,42
$\mathcal{B}$	Matrix element	79
C	Thermal speed	23
c	Speed of light	46
$C^N S^M$, $\mathcal{C}_n$	Expansion functions	64
$\mathcal{D}$	Matrix element	79
e	Electronic charge	29
f, f_o, $f^{(o)}$	Distribution functions, (total, local, and absolute maxwellians)	21,31,23
$\mathcal{F}$	Matrix element	79
F	Linear operator	58

Symbol	Description	
g	Perturbation distribution	21
h, h_1, h_o, $\mathscr{h}$	Inhomogeneous terms in Boltzmann equation	43,41,120
I_ℓ	Moment integral	49
J	Current	21
L, $\tilde{L}$	Linear operators, Larmor radius	56,50
$\mathscr{L}$	Matrix element	79
m, M, $\mathscr{m}$	Mass, (individual, total, reduced)	23,24,31
n, N	Number density, (component, total)	21
p, P	Pressure, (component, total)	21
$\wp$	Frequency ($\wp$ = ck)	91
Q	Charge density	21
R_{ij}	J-Flux tensor	36
$\mathscr{S}_n$	Expansion function	64
$\mathscr{T}$	Temperature term	41
T	Temperature	21
u, U	Velocity, (component, mass averaged)	21
w, W	Internal energy, (component, total)	21
X_i	System vector	46
x	Expansion parameter	63
y	Expansion parameter	63
Z, z	Heat flow vector, (total, component)	27
k	Wave number vector	42
α	Angle variable	43

Symbol	Description	Page
β, β_1	Velocity ratios	99,100
η_N	Magnetic coefficient	44
γ	Mass ratio	92
δ_{ij}	Kronecker Delta	
ε_{ijk}	Levi Civita symbol	
κ	Boltzmann's constant	
ν, ν_1, ν_2	Collision frequencies	31
ν_3, $\overline{\nu}_3$	Frequency; constant	45
ξ	Microscopic valocity	23
ρ, ϱ	Mass density, (component, total)	21
τ	Magnetic coefficient	59
ϕ	Exponential function	43
ψ	Angle integration variable	45
$\overline{\omega}$	Dissipative frequency	92
ω	Frequency of disturbance	42
ω_o	Plasma frequency (including reduced mass)	36
ϕ_o	Plasma constant	48
Ω, $\overline{\Omega}$, Ω, Ω^*, Ω'	Larmor frequencies	45,92,92,48,100a
ζ	Heat flow vector	27
λ_c	Mean free path	118

Propagation of Long Wavelength Disturbances in a Plasma

A. Introduction

The problem of uncovering the modes of excitation of a plasma in a uniform magnetic field has been extensively studied in the literature. For the most part these endeavors were constructed about a macroscopic, phenomenological theory. In addition, one is able to find many problems which yield solutions under special constraints, such as on the direction of propagation of the disturbances; statements pertaining to the vanishing of certain macroscopic variables (e.g., low temperature approximation, charge neutral assumption, etc.) and ad hoc assumptions relating the higher moments of the distribution function to its lower moments.

These investigations lead to distinct categories, e.g., magnetohydrodynamics,[1,2,3,4] cooperative phenomena[5] (plasma oscillations), magnetic resonance,[6,7,8] cold plasma theory,[9,10] collisionless limit (Vlasov equation),[11,12] etc.

In this analysis we have examined the general problem of uncovering the modes of development of a plasma in a uniform magnetic field. The plasma is displaced from an equilibrium state which is described by two absolute Maxwellians with equal temperatures, equal number densities, and zero macroscopic velocities, together with constant magnetic field and zero electric field. The state of the system is subsequently described by the perturbation of these variables away from equilibrium.

The principle concern has been to present a self-contained approach to the problem, starting from microscopic origins, and in this manner to: (1) find all the modes of excitation, (2) exhibit a formal ordering of phenomena into distinct categories, (3) uncover the connections between these modes, especially with respect to the collapse of entirely different domains to a common class of phenomena, and (4) follow the transition from microscopic to the macroscopic description.

We are primarily interested in long wavelength disturbances, i.e., the technique of solution involves expansions about small wave number. The equations of motion are two Boltzmann equations and Maxwell's equations which are coupled through the charge density and the current. These equations are perturbed away from the above stated equilibrium state, and the resulting system is linearized in the perturbation quantities. The linearized collision integrals are replaced by forms which yield the standard macroscopic equations, and are consistent with basic relaxation arguments. In addition these forms are shown to follow quite naturally from an expansion of the linearized collision integrals in terms of their own eigenfunctions, or through an itteration procedure of the full Boltzmann equations about local Maxwellian states, (cf. Appendix A). To uncover the subsequent development of the system a space-time Fourier-Laplace transform is introduced which immediately permits the distribution function to be written as a linear functional over the first five moments of both gases (viz., $\hat{n},n$; $\hat{u},u$; $\hat{T},T$).[13,14,15] If one then operates on the distribution function written in this manner with the related first five moment integral operators, one obtains a linear homogeneous set of algebraic equations. These equations, together with Maxwell's equations, which relate the electric field to the current and charge density, constitute a self-consistent set of equations in the thirteen transform variables which are the transforms of the two number densities, the two macroscopic velocities, the two temperatures, and the electric field. The coefficient matrix is a transcendental function of the frequency ω and wavelength k^{-1} of the perturbed state. For the most part, the relations we find are asymptotic to the exact form. A non-trivial state will exist only for those values of k and ω which make this matrix singular. It follows that the problem of uncovering the modes of development is essentially an eigenvalue problem. The related 13×1 column vector determines the state of the system. For convenience of reference we will call the said matrix the master matrix of this problem and the determinant will similarly be referred to as the master determinant.

The overall analysis is in two stages. Firstly, the above stated formal expansions are performed. In this procedure the parameter ck (c is the speed of light) is treated as an independent variable. Specifically, the limit $k \longrightarrow 0$ is taken along the path $ck = \text{constant}$. To terms of order k the master determinant is 14^{th} order in ω. These roots separate into two groups which are categorized in terms of the behavior of the roots about $k = 0$. In the second stage, the inversion equations in (x,t) space are examined in order to discover the domain of relevance of the related classes of phenomena. In this manner we find that the PEM (Plasma Electromagnetic) limit is characteristic of a low temperature plasma, and, furthermore, that within this limit the variation of number density generates an entire spectrum of phenomena, ranging from a coupling to pressureless MHD* (Magnetohydrodynamics plus displacement current) in the infinitely dense region to the vacuum electromagnetic and pure Larmor roots at the opposite extreme.

The generalized macroscopic equations which emerge (cf. Eqs. C-IV, 1-10) exhibit the following properties: (1) The momentum equations contain non-local-in-time pressure terms, which in the limit of large collision frequency reduce to standard forms. (2) The source terms of the momentum equations include difference in temperature contributions, in addition to difference in flow terms suggested by previous authors.[16,17,18] (3) The energy equations contain non-isotropic heat conductivities which are functions of the temperatures of both gases. In the limit of equal masses and temperatures and large collision frequencies these forms reduce to the classical Chapman-Cowling result.[19] (4) The energy equations contain difference in temperature sources which contribute to the time rate of change of the individual entropies, so that to lowest order the component gases are non-adiabatic.

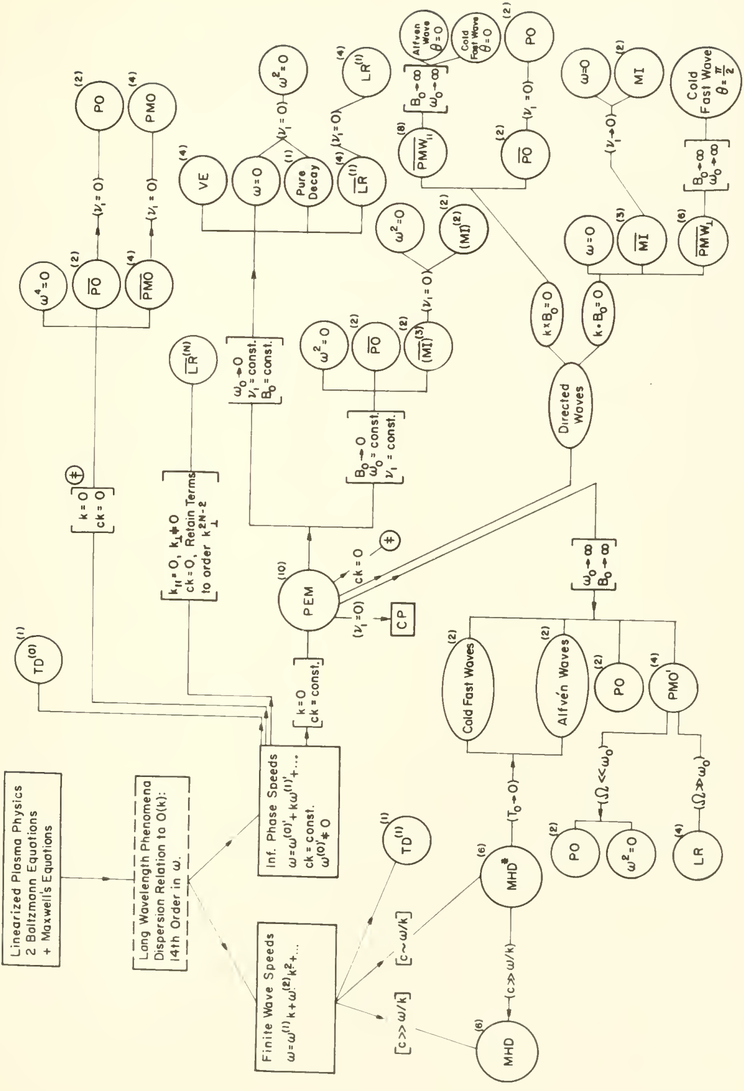
The classification of phenomena into two groups as mentioned above is specified according to the vanishing or non-vanishing of the leading term in the expansion of the dispersion relation about $k = 0$, viz., $\omega = \omega^{(0)} + k\omega^{(1)} + \ldots$. The roots of the former class are indicative of finite wave speed phenomena (MHD), while those of the later class (PEM) include infinite phase speed phenomena. In the special case of vanishing collision frequency, the PEM region reduces to the cold plasma domain.[7,8,20] This terminology is motivated by the observation that the relevant momentum equation follows from the more general one in the limit of vanishing temperature, i.e., $[\kappa T_o/MCv] << 1$, where κ is Boltzmann's constant, M is the mass, C is the thermal speed, and v is the speed ω/k. In the PEM region $v \rightarrow \infty$, as $k \rightarrow 0$ and the inequality is naturally established. A similar observation is evident with respect to the MHD region wherein one notices that the relevant current equation follows from the more general one in the limit of large number density, or, more precisely: $\omega_o^2 >> \omega^2$, where ω_o is the plasma frequency, and ω is the frequency of the disturbance. In the MHD region, $\omega \rightarrow 0$ as $k \rightarrow 0$, and this requirment is automatically satisfied.

The dispersion relation of MHD will be shown to be 6^{th} order in ω, while the dispersion relation of PEM will be seen to be 10^{th} order in ω. Including the two distinct temperature decays, this complement of modes includes eighteen roots. Since the original master determinant is 14^{th} order in ω, it follows that four of the six MHD roots and four of the ten PEM roots are different forms of the same modes. These four common modes are shown to be the doubly degenerate cold fast waves and the two Alfven waves.

The diagram below indicates the various subdivisions of phenomena which are induced according to the above scheme. The superscript in parenthesis outside of a circle indicates the order of the included root, e.g., $\textcircled{\scriptsize VE}^{(4)}$ represents the Vacuum Electrodynamic roots $(\omega^2 - k^2c^2)^2 = 0$. A bar over a classification symbol means that dissipative effects are included. The star in MHD^* indicates the inclusion of the displacement current. The symbol $\mathrm{LR}^{(N)}$ is indicative of the N^{th} order Larmor resonance, viz., $\omega \sim N\Omega + \ldots$ (N = integer, Ω = Larmor frequency).

The first relevant observation is that the PEM class of phenomena exhibits a temperature decay (TD) to $\Delta T = 0$, with a decay time: $(2\nu_2)^{-1}$. The superscripts (0) and (2) on the TD symbol indicate that in the PEM domain the displacement from $\Delta T = 0$ is a zeroth order effect in k while in the MHD domain it is a second order effect in k. If we look ahead to the matrix formulation of the problem for a moment, then it is quite evident that to lowest order the master matrix is diagonal with a 2×2 block multiplying the two temperature entries in the system vector. It follows that in the PEM case, the system eigenvector which corresponds to the oscillatory m modes includes $T = \hat{T} = 0$. The complementary eigenvector includes a temperature decay along the path $\hat{T} + T = 0$, i.e., the perturbation temperatures are displaced equally and oppositely.

The junction between MHD and MHD^* indicates that the effects of the displacement current become negligible for waves whose


Linearized Plasma Physics
2 Boltzmann Equations
+ Maxwell's Equations
Long Wavelength Phenomena
Dispersion Relation to O(k):
14th Order in ω.
Inf. Phase Speeds
ω=ω(0)'+kω(1)'+...
ck = const.
ω(0)' ≠ 0
Finite Wave Speeds
ω=ω(1)k+ω(2)k²+...
TD(0) (1)
TD(1) (1)
[c>>ω/k]
[c~ω/k]
MHD (6)
MHD* (6)
(c>>ω/k)
(T0→0)
Cold Fast Waves (2)
Alfvén Waves (2)
PO (2)
PMO' (4)
(Ω<<ω0)
(Ω>>ω0)
PO (2)
ω²=0
LR (4)
[k=0, ck=const.]
PEM (10)
(ν1=0)
CP
ck=0
≠
[k=0, ck=0] ≠
ω⁴=0
PO (2)
PMO (4)
(ν1=0)
PO (2)
PMO (4)
[k∥=0, k⊥≠0, ck=0, Retain Terms to order k⊥^(2N-2)]
LR(N)
[ω0→0, ν1=const., B0=const.]
VE (4)
ω=0
Pure Decay (1)
LR(1) (4)
(ν1=0)
ω²=0
(ν1=0)
LR(1) (4)
[B0→0, ω0=const., ν1=const.]
ω²=0
PO (2)
(MI)(3) (2)
(ν1=0)
ω²=0
(MI)(2) (2)
Directed Waves
k×B0=0
k·B0=0
PMW∥ (8)
[B0→∞, ω0→∞]
Alfven Wave θ=0
Cold Fast Wave θ=0
PO (2)
(ν1=0)
PO (2)
ω=0
MI (3)
(ν1→0)
ω=0
MI (2)
PMW⊥ (6)
[B0→∞, ω0→∞]
Cold Fast Wave θ=π/2
[ω0→∞, B0→∞]

velocities are much less than the speed of light, c. The formalism subsequently presented indicates that the lowest order form of the current equation is the classical $E + U \times B_0 = 0$. If these roots were extended to next order in k, both the dissipation, $-\nu_1 J$ and the Hall current term, $\bar{\Omega} J \times B$ (cf. Eq. B-II, 23) would appear, in addition to the latter form. However, most investigators[20,21,22] disregard the $J \times B$ term when examining these dissipative effects. Such results (i.e., excluding the Hall current) would follow from the more general theory in the limit of large number density.

The second region of phenomena is governed by the dispersion relation $\omega = \omega^{(0)} + k\omega^{(1)} + \ldots, \omega^{(0)} \neq 0$. As already noted, the parameter ck which enters from the Maxwell's equations part of the system is treated as an independent parameter, which accounts for the presence of k in the lowest order dispersion relations.

Pure oscillations result in the limit $ck \rightarrow 0$, together with $k \rightarrow 0$. This limit gives rise to decaying plasma oscillations $(\overline{PC}^{(2)})$, and decaying plasma magnetic oscillations $(\overline{PMO}^{(2)})$. These roots are identical to those found by Grad[23] who demonstrates directly that the plasma oscillations are associated with current fluctuations parallel to B_0, while the plasma magnetic oscillations are associated with current fluctuations perpendicular to B_0. In general, in the following limits wherein $B_0 \neq 0$, those modes which do not contain the magnetic field are the eigenvalues of the eigenvector $(J_{\parallel}, 0)$, while the complementary non-degenerate roots are the eigenvalues of $(0, J_\perp)$, where J is the current, and directions are with respect to B_o. The remaining fourth order $\omega^4 = 0$, is the degeneracy of the vacuum electrodynamic roots $(\omega^2 - k^2c^2)^2 = 0$, in the above limit.

The problem of Larmor resonances has been estensively covered in the literature.[6,7,24] However, most authors, when investigating this problem, impose the constraint of stationary ions and do not include the effect of collisions. In this vein, the dispersion of waves directed normal to the B_o axis was investigated. The lowest order non-trivial dispersion of the N^{th} order resonance includes terms of order $k_\perp^{2N-2}$. The effect of the self-collisions alters the zeroth order $(k = 0)$ results, wherein the decay time is as the inverse self-collision frequency ν, and is independent of the resonance order of the mode. More generally, the dispersion was seen to vanish in the limit of large magnetic field or vanishing number density. These Larmor resonances are seen to constitute a doubly degenerate four fold infinity of roots, two about every multiple Larmor frequency of both species (cf. footnote p. 50).

In the more general case kc = constant, $k = 0$, one obtains the lowest order propagating PEM phenomena. The governing equations for this region include a full equation of motion for the current as opposed to the simplified $E + U \times B_o = 0$ or MHD. However, as already observed, in the limit of large number density (i.e., plasma frequency large), the more general equation of motion for the current collapses to the above form. This again gives substance to the idea that the Lundquist equations govern a fluid dynamical medium, since the relevant equations follow from the more general ones in the limit of large number densities. In a similar manner, the more general momentum equation of MHD (cf. Eq. C-IV, 12) collapses to the simpler momentum equation of PEM (cf. Eq. C-IV, 29) in the limit of vanishing temperature. In these connecting limits the sound speed is obviously surpressed, so that the fast wave of MHD becomes the isotropic wave, $v^2 = A'^2$ (cold fast wave), while the Alfven waves remain intact. The remaining roots in this limit of large number density are the plasma oscillations ($PO^{(2)}$) and four plasma magnetic oscillations (PMO'), (cf. Eq. C-IV, 56). If the plasma frequency is increased far beyond the Larmor frequency, then these roots to to a degenerate $\omega^2 = 0$, together with the two plasma oscillations, $\omega^2 = \omega_o^2$. In the opposite extreme of the Larmor frequencies far exceeding the plasma frequency, these plasma magnetic oscillations reduce to the four pure Larmor modes ($LR^{(4)}$).

The effect of decreasing the large number density in this region was uncovered by extending the related calculation to terms of order $[kc/\omega_o]^4$. In the direction parallel to the magnetic field the speed of the cold fast wave becomes separated from that of the Alfven wave by the amount $(kc/\omega_o)((\hat{m}-m)/\sqrt{\hat{m}\,m})$, where $\hat{m}$ and m are the ion and electron masses respectively. In the normal direction the velocity of the cold fast wave is decreased by the factor $(1-(k^2c^2/\omega_o^2))$.

The first directed wave ($k \times B_o = 0$) yields eight decaying plasma magnetic waves $(\overline{PMW}_{\parallel})^{(8)}$ (cf. Eq. C-IV, 64) and two decaying plasma oscillations. These waves are the extension of the classical non-dissipative waves reported by Bernstein and Trehan.* In the limit of large number density the magnetic field both the cold fast waves and Alfven waves are recaptured. Spitzer** has discussed these roots in the limit of large mass ratio in which case it is impossible to recapture the correct MHD roots. This large mass ratio limit is also reported by Oster† who recalls that they are the ordinary (+) and extra-ordinary (-) modes of ionosphere theory.

The second directed wave ($k \cdot B_o = 0$) (cf. Eq. C-IV, 69) again gives the extension of the non-dissipative waves reported by Bernstein and Trehan††. The six plasma-magnetic waves $(\overline{PMW}_{\perp})^{(6)}$ in the limit of large number density and large magnetic field yield the cold fast waves in the direction $\theta = \pi/2$. The decaying magneto-ionic roots $(\overline{MI}^{(3)})$ are a generalization of the non-dissipative magneto-ionic roots.[25] The root that degenerates to $\omega = 0$ when the dissipation is set equal to zero behaves as: $i\omega \sim \nu_1 c^2 k^2/(\omega_o^2 + c^2 k^2)$ (cf. Eq. C-IV, 44.1), for small collision frequency ν_1.

---

* Cf. Ref. 20, Eq. III-71.

** Cf. Ref. 16, Eq. 4-13.

† Cf. Ref. 25, Eq. 131.

†† Cr. Ref. 20, Eq. III-82.

For waves propagating in an arbitrary direction, two limiting cases were considered.* These are vanishing number density and, secondly, vanishing magnetic field. In the first limit (cf. Eq. C-IV, 39), as expected, the vacuum electromagnetic roots ($VE^{(4)}$) are recaptured. Besides a degenerate $\omega^2 = 0$, the remaining modes are a purely decaying mode and four decaying Larmor modes ($\overline{LR}^{(1)}$).[4] (The superscript (1) is indicative of first order resonance.)

In the remaining limit of vanishing magnetic field the decaying plasma oscillations and decaying magneto-ionic waves once more appear, in addition to the doubly degenerate $\omega^2 = 0$.

The analysis which is to follow suggests a connection between the so called macroscopic and microscopic approaches. The distinction between these two regions is that the former domain is governed by the distribution functions, while the latter is characterized by the less detailed moments of the distributions. In the present formulation the distribution function appears as a linear functional of the first three moments of both gases. This functional is expanded about small k. The more terms that are retained, the more precise is the resulting approximation to the distributions. At any point in the procedure macroscopic equations may be obtained by forming the proper inversions back to (x,t) space. The order of these differential equations relates to the order of k kept in the expansions. In this sense the detailed fluctuations of the distributions are characterized by the higher order derivatives in the macroscopic equations. A good example of this connection is in the magnetic resonance problem, wherein the dispersion of the N^{th} resonance is of order k^{2N-2} so that the macroscopic equations of order $2N-2$ exhibit resonances up to order N.

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* Since there are only three external parameters in the problem, B_0, n_0, T_0; and T_0 is already constrained, there remain only two relevant extreme possibilities.

B. Equations and Linearization.

I. Linearization of variables.

Before wading into the formulation, perhaps it is wise at this point to impart a note on the notation. To distinguish between ion quantities and electron quantities we adopt the convention that ion variables carry a roof and electron variables are bare. For instance, the two macroscopic velocities are written $\hat{u}$ and u; the former for ions and the letter for electrons. To distinguish between full variables and perturbation quantities we put a bar on top of the full variable and nothing on the perturbed variable. For instance, the perturbation of the electron temperature is described by the equation $\overline{T} = T_o + T$. All variables that make up the initial equilibrium situation carry a subscript zero which explains T_o in the latter equation.

The linearization is about an absolute Maxwellian $f^{(o)}$ and obeys the following scheme:

$$f = f^{(o)} + g \tag{1}$$

$$\begin{aligned}
&\overline{E} = 0 + E, \quad \overline{n} = n_o + n, \quad \overline{u} = 0 + u, \quad \overline{p}_{ij} = p_o^{ij} + p_{ij} \\
&\overline{B} = B_o + B, \quad \overline{T} = T_o + T, \quad \overline{U} = 0 + U, \quad \overline{\zeta} = \zeta_o + \zeta \\
&\overline{w} = w_o + w, \quad \overline{\rho} = \rho_o + \rho, \\
&\overline{J} = 0 + J, \quad \overline{Q} = 0 + Q, \quad \rho = \rho_o + \rho, \quad \overline{N} = N_o + N \\
&\overline{P}_{ij} = P_o^{ij} + P_{ij}, \quad \overline{Z} = Z_o + Z, \quad \overline{W} = W_o + W \, .
\end{aligned} \tag{2}$$

Except for the individual temperatures T, $\hat{\overline{T}}$, capital letters denote combined fluid variables. They are related to the lower case letters (individual flow fariables) in a purely additive manner, i.e., $\overline{A} = \overline{a} + \hat{\overline{a}}$ (A represents ρ, P, N, Z, W) except for the mean flow U which is given by:

$$\overline{\rho U} = \overline{\rho u} + \overline{\hat{\rho}\hat{u}}. \tag{3}$$

In the above set (2), E is electric field intensity, B is magnetic field intensity, w is internal energy, n is number density, ρ is mass density, u is macroscopic velocity, p is pressure, ζ is heat flow, J is current density, and Q is charge density. It should be noted that to any individual flow variable, (lower case letter) which does not carry a roof (i.e., electron variables) there corresponds a similar symbol which is roofed (ion variable). An example is $\hat{u}$ which is the said "reflected" u variable. In this manner there are eight more equations in the set (1,2) which are so generated.

The equilibrium configuration is further defined by:

$$J_o = 0 = Q_o$$

(4)
$$U_o = u_o = \hat{u}_o = 0, \quad n_o = \hat{n}_o = N_o/2, \quad \rho_o = mn_o + \hat{m}n_o = \frac{N_oM}{2}$$

$$w_o = \frac{3}{2} n_okT_o, \quad \hat{w}_o = \frac{3}{2} \hat{n}_ok\hat{T}_o, \quad T_o = \hat{T}_o, \quad \text{so that}$$

$$w_o + \hat{w}_o = \frac{3}{2} N_okT_o = W_o , \quad w_o = \hat{w}_o = W_o/2 .$$

It is convenient at this point to introduce the scalar pressure $\overline{p}$,

(5)
$$\overline{p} = \frac{1}{3} \mathrm{Tr}\, \overline{p}_{ij} = \frac{2}{3} \overline{w}.$$

The notation is such that any p without subscripts is a scalar pressure.

This latter identity symbol follows from the kinetic definition of pressure and internal energy (to be introduced momentarily).

The stress s_{ij} is defined as the traceless part of the pressure tensor, viz.,

(6)
$$\overline{s}_{ij} = \overline{p}_{ij} - \overline{p} .$$

For the equilibrium scalar pressure we write

$$p_o + \hat{p}_o = P_o \,, \qquad p_o = \hat{p}_o = P_o/2 = n_o k T_o \tag{7}$$

$$P_o = N_o k T_o \;.$$

The absolute Maxwellian $f^{(o)}$ is about zero mean velocity and is given by

$$f^{(o)} = \frac{n_o}{(2\pi)^{3/2} C^3} \exp\left\{ -\xi^2/2C^2 \right\} = \frac{n_o}{C^3}\,\phi \;, \tag{8}$$

which also defines the exponsntial function ϕ. The particle velocity is ξ, and the thermal speed C is defined through the equilibrium temperature T_o, viz.,

$$mC^2 = kT_o = \hat{m}\hat{C}^2 . \tag{9}$$

The electron and ion masses are m and $\hat{m}$ respectively.

Let us now formulate expressions for these perturbation quantities in terms of velocity moments of the perturbation distribution function g. Firstly, for the number density we recall the kinetic definition:

$$n = \int f \, d^3\xi = \int (f^{(o)} + g) d^3\xi = n_o + n \tag{10}$$

so that

$$n = \int g \; d^3\xi . \tag{11}$$

The mass density is related to the number density as

$$\bar{\rho} = \bar{n}m + \hat{\bar{n}}\hat{m}$$

(12) $$= n_o m + \hat{n}_o \hat{m} + nm + \hat{n}\hat{m}$$

$$= \rho_o + \rho = \frac{N_o M}{2} + \rho$$

(12') $$M = \hat{m} + m,$$

so that the perturbation mass density is given by:

(13) $$\rho = nm + \hat{n}\hat{m} \; .$$

For the velocity $\bar{u}$ we recall

(14) $$\bar{n}\bar{u} = \int \xi f d^3\xi = \int \xi (f^{(o)} + g) d^3\xi$$

(15) $$\bar{n}\bar{u} = 0 + n_o u \; ,$$

so that,

(16) $$n_o u = \int \xi g d^3\xi \; , \quad \text{and} \quad \rho_o u = m \int \xi g d^3\xi .$$

It follows that for the mass averaged flow U, we have

(17) $$\bar{U} = 0 + U$$

and

(18) $$\rho_o U = \rho_o u + \hat{\rho}_o \hat{u} \; .$$

The internal energy is defined in terms of the kinetic energy measured in a frame which is moving with the velocity $\bar{U}$, so that

$$W = \frac{1}{2} m_i \int f_i(\xi-\overline{U})^2 d^3\xi . \tag{19}$$

The sum over repeated indices covers ions and electrons. The linearization yields:

$$\begin{aligned} \overline{W} &= W_o + W = \frac{1}{2} m_i \int f_i^{(o)} \xi^3 d^3\xi + \frac{1}{2} m_i \int g_i \xi^2 d^3\xi , \\ W &= w + \hat{w} = \frac{1}{2} m_i \int g_i \xi^2 d^3\xi \end{aligned} \tag{20}$$

$$w = \frac{1}{2} m \int g \xi^2 d^3\xi , \qquad w_o = \frac{3}{2} n_o \kappa T_o . \tag{21}$$

We recall that the temperature is related to the internal energy through the kinetic definition of temperature (internal energy per particle):

$$\overline{w} = \frac{3}{2} \overline{n} \kappa \overline{T} , \tag{22}$$

the linearization of which gives

$$\begin{aligned} n\kappa T &= \kappa(n_o + n)(T_o + T) = \kappa n_o T_o + \kappa(nT_o + n_o T) \\ &= \frac{1}{3} m \int \xi^2 f = \frac{1}{3} m \int \xi^2 (f^{(o)} + g), \end{aligned} \tag{23}$$

whence it follows that

$$\frac{1}{3} m \int \xi^2 g = \kappa \int g T_o + n_o T \kappa . \tag{24}$$

In terms of the thermal speed $c^2 = kT/m$ this latter equation appears as

$$\frac{n_o T C^2}{T_o} = \frac{1}{3} \int g(\xi^2 - 3C^2)d^3\xi \ , \tag{25}$$

the non-dimensional form of which appears as

$$T = \frac{1}{3} \int g(\xi^2 - 3)d^3\xi \ . \tag{26}$$

If the subscript d denotes "dimensional", then $T_d = T_o T$, $g_d = \frac{n_o}{C^3} g$, $\xi_d = \xi/C$. We will return to the non-dimensional formulation when discussing the Boltzmann equations.

The pressure is defined in terms of momentum transport measured in a frame moving with the velocity $\overline{U}$. We write accordingly

$$\overline{P}_{\ell n} = m_i \int f_i(\xi - \overline{U})_\ell (\xi - \overline{U})_n \ . \tag{27}$$

Linearization yields

$$\begin{aligned} P_{\ell n} &= m_i \int f_i^{(o)} \xi_\ell \xi_n + m_i \int g_i \xi_\ell \xi_n \\ &= P_o + P_{\ell n} \ . \end{aligned} \tag{28}$$

Note:

$$\begin{aligned} P_o &= \frac{1}{3} \operatorname{Tr} m_i \int f_i^{(o)} \xi_\ell \xi_n = \frac{1}{3} \operatorname{Tr} P_o^{\ell n} \\ P_{\ell n} &= m_i \int g_i \xi_\ell \xi_n = \hat{p}_{\ell n} + p_{\ell n} \ . \end{aligned} \tag{29}$$

Similarly, the heat flow vector Z is the heat flow measured in the $\overline{U}$ frame.

(30) $$\overline{Z} = \frac{1}{2} m_i \int f_i (\xi - \overline{U})^2 (\xi - \overline{U}) \ .$$

Linearization yields .

(31) $$\overline{Z} = -U \frac{1}{2} m_i \int f_i^{(o)} (\xi^2 + \int \xi_\ell^2) + \frac{1}{2} m_i \int g_i \xi^2 \xi$$

$$\overline{Z} = 0 - U \frac{5}{2} (N_o \kappa T_o) + \frac{1}{2} m_i \int g_i \xi^2 \xi$$

and for the individual heat flow we write:

(32) $$\overline{\zeta} = 0 - U \frac{5}{2} n_o \kappa T_o + \frac{1}{2} m \int g \xi^2 \xi \ .$$

It follows that the perturbation quantities are given by

(33) $$Z = - \frac{5}{3} U W_o + \frac{1}{2} m_i \int g_i \xi^2 \xi \ ,$$

and for the individual perturbation heat flows:

(34) $$\zeta = - \frac{5}{3} U w_o + \frac{1}{2} \int m g \xi^2 \xi \ .$$

It is proper at this point to introduce a less physically significant heat flow ζ', the energy transport measured in the individual frames of each gas. This heat flow is generated from the above formulation by letting $U \longrightarrow u$ for each component, so that ζ' appears as

(35)
$$\zeta' = -\frac{5}{3}\, u w_o + \frac{1}{2}\, m \int g \xi^2 \xi$$

$$\zeta' = \frac{m}{2} \int g(\xi^2 - 5C)\xi d^3\xi .$$

It follows that the difference between ζ and ζ' is

(36)
$$\zeta = \zeta' - \frac{w_o \hat{m}}{\hat{m}+m}\, (\hat{u}-u)$$

$$\hat{\zeta} = \hat{\zeta}' + \frac{w_o m}{\hat{m}+m}\, (\hat{u}-u)$$

so that the difference in the total heat flows is

(37)
$$Z = Z' - w_o\, \frac{\hat{m}-m}{\hat{m}+m}\, (\hat{u}-u) .$$

These two heat flows will be the same if either the masses of the two components are equal or the velocities of the components are equal.

In the non-dimensional units introduced above, ζ' appears as

(38)
$$\zeta' = \frac{1}{2}\, m n_o C^3 \int g(\xi^2 - 5)\xi d^3\xi .$$

The charge density $\overline{Q}$ is given by:

(39)
$$\overline{Q} = e(\hat{\bar{n}} - \bar{n})$$

$$\overline{Q} = 0 + e(\hat{n}-n),$$

so that the perturbation charge density is written

(40)
$$Q = e(\hat{n}-n).$$

The current density J is written

$$\begin{aligned} \bar{J} &= e(\hat{\bar{n}}\hat{\bar{u}} - \bar{n}\bar{u}) \\ \bar{J} &= 0 + en_o(\hat{u}-u) \ , \end{aligned} \tag{41}$$

whence the perturbation is given by

$$J = en_o(\hat{u}-u). \tag{42}$$

Before leaving this section on the formulation of the perturbation variables, we note for future reference, the following relation between the individual fluid variables $[n,\hat{n};\ u,\hat{u}]$, and the total fluid variables $[\rho,Q;\ U,J]$. First we repeat the above derived relations.

$$\begin{aligned} \rho &= \hat{m}\hat{n} + mn \qquad & J &= en_o(\hat{u}-u) \\ Q &= e(\hat{n}-n) \qquad & U &= \frac{\hat{m}\hat{u} + mu}{\hat{m} + m} \end{aligned} \tag{43}$$

and secondly, the inverses

$$\begin{aligned} n &= \frac{1}{M}\left(\rho - \frac{\hat{m}}{e}Q\right) \qquad & \hat{u} &= U + \frac{m}{en_oM}J \\ \hat{n} &= \frac{1}{M}\left(\rho + \frac{m}{e}Q\right) \qquad & u &= U - \frac{\hat{m}}{en_oM}J \ . \end{aligned} \tag{44}$$

B-II. Boltzmann and Moment Equations.

In Appendix A we indicate the manner in which the perturbation away from the above said equilibrium state, i.e., $[\hat{f},f,E,B]_{(\mathrm{equil})} = [\hat{f}^{(o)}, f^{(o)}; 0, B_o]$, and subsequent linearization yields the following linearized Boltzmann equations.

$$\nu g + \frac{\partial g}{\partial t} + \xi \cdot \nabla g - \frac{e}{m} \xi \times B_o \cdot \nabla_\xi g - \frac{e}{m} E \cdot \nabla_\xi f^{(o)} = f^{(o)}[\nu \sum_{n=0}^{2} a^{(n)} H^{(n)} + \frac{\nu_1 \mathfrak{m}}{m} H^{(1)} \cdot \Delta u + \nu_2 H^{(2)} \Delta T] \tag{1}$$

$$\hat{\nu}\hat{g} + \frac{\partial \hat{g}}{\partial t} + \xi \cdot \nabla \hat{g} + \frac{e}{\hat{m}} \xi \times B_o \cdot \nabla_\xi \hat{g} + \frac{e}{\hat{m}} E \cdot \nabla_\xi \hat{f}^{(o)} = \hat{f}^{(o)}[\hat{\nu} \sum_{n=0}^{2} \hat{a}^{(n)} \hat{H}^{(n)} - \frac{\nu_1 \mathfrak{m}}{\hat{m}} \hat{H}^{(1)} \cdot \Delta u - \nu_2 H^{(2)} \Delta T]. \tag{2}$$

The $H^{(n)}$ Hermite polynomials are generated from a local Maxwellian[13,14] in the following manner. If the local Maxwellian f_o

$$f_o = \frac{\bar{n}}{(2\pi R\bar{T})^{3/2}} \exp\left[- \frac{(\xi - \bar{u})^2}{2R\bar{T}}\right] , \tag{3}$$

is expanded about the equilibrium configuration, linearization yields

$$f_o = f^{(o)}[1 + \frac{n}{n_o} H^{(o)} + \frac{u}{C} \cdot H^{(1)} + \frac{T}{T_o} H^{(2)}] \tag{4}$$

where

$$H^{o} = 1; \quad H^{(1)} = \frac{\xi}{C}; \quad H^{(2)} = \frac{1}{2}[(\frac{\xi}{C})^2 - 3] . \tag{5}$$

$\hat{H}^{(n)}$ accordingly is a function of $\xi/\hat{C}$. The coefficients $a^{(n)}$ are: $a^{(o)} = n/n_o$; $a^{(1)} = u/C$; $a^{(2)} = T/T_o$.

The variable $\mathcal{M}$ is the reduced mass $m\hat{m}/(m+\hat{m})$. We note that the model includes four dissipation constants, viz., $\nu, \hat{\nu}, \nu_1, \nu_2$. The first two are related to the self-collision frequency of the Krook-Bhatnager-Gross[15] model and are associated with the time of relaxation of the gas to a local Maxwellian state. The frequency ν_1 emerges as being related to the electrical conductivity, and is associated with a relaxation from a state of flow of one gas through the other to a quiescent state. The third dissipation frequency ν_2 induces an equilibrium state wherein both gases have the same temperature, i.e., the same energy per particle.

These constants are positive so as to insure relaxation in the correct direction in time. In addition, the specific orientation of the mass parameters will be seen to yield the correct conservation equations.

Let us here recall the integral forms of the flow variables derived above:

$$n = \int g d^3\xi \ , \quad n_o u = \int \xi g d^3\xi \ , \quad w = \frac{m}{2} \int g \xi^2 d^3\xi$$

$$(6) \qquad n_o(T/T_o) = \frac{1}{3} \int g\left(\left(\frac{\xi}{C}\right)^2 - 3\right) d^3\xi \ , \quad p_{\ell n} = m \int g \xi_\ell \xi_n d^3\xi$$

$$\zeta' = \frac{m}{2} \int g(\xi^2 - 5C^2)\xi d^3\xi \ ,$$

which also serves to generate orthogonal polynomials with respect to the Maxwellian weighting factor $f^{(o)}$. To preserve orthogonality of these polynomials, the temperature is taken as the third moment instead of internal energy.*

Returning to equations (1), we consider the three operations: $\int d^3\xi$, $\int d^3\xi m\xi$, $\int d^3\xi \frac{m}{2}\xi^2$, to obtain

(a) The continuity equations:

$$\frac{\partial n}{\partial t} + n_o \nabla\cdot u = 0 \tag{7}$$

$$\frac{\partial \hat{n}}{\partial t} + n_o \nabla\cdot\hat{u} = 0, \tag{8}$$

where we have recalled that $\nabla_\xi f^{(o)} = -\xi f^{(o)}/C^2$ and

$$\xi\times B\cdot\nabla_\xi g = \nabla_\xi\cdot\xi\times Bg \ . \tag{9}$$

(b) The momentum equations:

When we operate with $\int \xi_i d^3\xi$ there remains

$$n_o \frac{\partial u_i}{\partial t} + \int d^3\xi\ \xi_i\xi_j \frac{\partial}{\partial x_j} g - \frac{e}{m}\int d^3\xi\ \xi_i \frac{\partial}{\partial \xi_j} g\varepsilon_{\ell nj}\xi_\ell B_n + \frac{e}{m} n_o E_i = \frac{\nu_1}{m} n_o \Delta u_i \tag{10}$$

$$\frac{\partial}{\partial x_j}\int d^3\xi\ \xi_i\xi_j g = \frac{\partial}{\partial x_j}\frac{p_{ij}}{m} \qquad \text{by Eq. (B-I: 29).} \tag{11}$$

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* The change is one of normalization. Instead of including the energy per unit volume, w, we discuss T, the energy per unit volume, per particle.

The $\xi \times B$ term is expanded as follows:

$$\varepsilon_{\ell nj} B_n \xi_i \frac{\partial}{\partial \xi_j} g\xi_\ell = \varepsilon_{\ell nj} B_n \left[\frac{\partial}{\partial \xi_j} g\xi_\ell \xi_i - g\xi_\ell \delta_{ij}\right]$$

$$\varepsilon_{\ell nj} B_n \xi_\ell \delta_{ij} = \varepsilon_{\ell ni} B_n \xi_\ell \ ,$$

which is the i^{th} component of $\xi \times B$, so that the above moment equation reduces to (here we include the ion equation)

$$(12) \qquad n_o \frac{\partial u}{\partial t} + \frac{e}{m} n_o (E + u \times B_o) + \frac{1}{m} \nabla \cdot p_{ij} = \frac{\mathcal{M}}{m} \nu_1 n_o \Delta u$$

$$(13) \qquad n_o \frac{\partial \hat{u}}{\partial t} - \frac{e}{\hat{m}} n_o (E + \hat{u} \times B_o) + \frac{1}{\hat{m}} \nabla \cdot \hat{p}_{ij} = - \frac{\mathcal{M}}{\hat{m}} \nu_1 n_o \Delta u \ .$$

Let us observe in passing that the formation of these moment equations supposes the vanishing of g at the boundaries of velocity space. It will be interesting to observe the location of this assumption in the analysis section to follow. One also notices that the Δu interaction terms indicate that the relative motion of the gases serves as a momentum source for the individual components.

(c) The energy equations:

To form the transfer equation for w we operate with $\frac{1}{2} m \int d^3\xi \xi^2$. First let us note the manner in which the self-collision terms cancel.

$$(14) \qquad \frac{1}{2} \nu m \int f^{(o)} \xi^2 \left(\frac{n}{n_o} + \frac{u \cdot \xi}{c^2} + \frac{T}{T_o} H^{(2)}\right) d^3\xi = \nu \frac{3}{2} k (T_o n + T n_o)$$

$$= \nu w = \frac{1}{2} m \nu \int g \xi^2 d^3\xi \quad ,$$

so that the ν terms cancel, and the only remaining collision term is the ΔT term.

The magnetic term includes the factor

$$(15) \qquad \int \xi^2 \nabla\cdot\xi \times Bg = \int \xi^2 \frac{\partial}{\partial\xi_\ell} \xi_n B_m \varepsilon_{nm\ell} g$$

$$= \int \frac{\partial}{\partial\xi_\ell} (\xi^2 \xi_n B_m \varepsilon_{nm\ell} g) - 2\varepsilon_{nm\ell} B_m \int g\xi_n \xi_\ell .$$

The first term vanishes because it is a surface term, while for the second term we note that the vector representation includes the factor $\xi \times B\cdot\xi$, which vanishes since $\xi \times B$ is normal to ξ.

Finally for the spacial part of the convective derivative we have, by equation (B-I: 34)

$$(16) \qquad \nabla\cdot \frac{1}{2} m \int g\xi^2\xi = \nabla\cdot(\zeta + \frac{5}{3} Uw_o) .$$

The energy equations appear as

$$(17) \qquad \frac{\partial}{\partial t} w + \frac{5}{3} w_o \nabla\cdot U + \nabla\cdot\zeta = \frac{3}{2} \nu_2 n_o k\Delta T$$

$$(18) \qquad \frac{\partial}{\partial t} \hat{w} + \frac{5}{3} w_o \nabla\cdot U + \nabla\cdot\hat{\zeta} = -\frac{3}{2} \nu_2 n_o k\Delta T .$$

Here we notice that a displacement in temperature between the two gases acts as an energy source for the individual components.

It is consistent at this point to form some linear combinations of the above equations (7,8; 12,13; 17,18) so as to obtain a more familiar set. For the most part, we are making the transformation of variables (cf. Eq. B-I: 43,44).

$$\begin{aligned} &n,\ \hat{n} \rightarrow \rho,\ Q \\ (19) \qquad & \\ &u,\ \hat{u} \rightarrow U,\ J\ . \end{aligned}$$

Mathematically, either system has its independent desirability, but physically the transformation is significant because any observations which are made in the laboratory are, indeed, with respect to the variables on the right. We note that the list is not extended to the higher moments (e.g., p_{ij}, ζ, ...). Although one readily agrees that the total components are physically relevant, e.g., $P_{ij} = p_{ij} + \hat{p}_{ij}$, it would appear that the related difference variables (with appropriate mass factors) have not come to the fore, so to speak. These combinations will suggest themselves immediately when we form the so called difference equations.

To obtain the system equivalent to (7,8) we multiply first by the respective masses and add. The second equation is formed by multiplying by the charge e and subtracting the ion equation from the electron equation. In this manner we obtain:

$$(20) \qquad \frac{\partial}{\partial t} \rho + \rho_o \nabla \cdot U = 0$$

$$(21) \qquad \frac{\partial}{\partial t} Q + \nabla \cdot J = 0.$$

In the exact same manner we form the system equivalent to the two momentum equations:

(22) $$\rho_o \frac{\partial}{\partial t} U + \nabla \cdot P_{ij} = J \times B_o$$

(23) $$\frac{\partial J}{\partial t} + \nabla \cdot R_{ij} - \omega_o^2 \, \varepsilon_o (E + U \times B_o) + \Omega \, J \times B = -\nu_1 J \; .$$

To obtain the second equation we noted that

(24) $$\hat{u}/\hat{m} + u/m = U/\mathcal{M} + \frac{J}{en_o} \, [m-\hat{m}/\hat{m} \; m]$$

(25) $$\mathcal{M} = m \; \hat{m}/(m+\hat{m}) \; .$$

In the second of these equations, ε_o is the dielectric constant, ω_o^2 is the plasma frequency,

$$\omega_o^2 = e^2 \, n_o / \varepsilon_o \mathcal{M}$$

measured with respect to the reduced mass $\mathcal{M}$. $\bar{\Omega}$ is a modified Larmor frequency,

(26) $$\bar{\Omega} = eB_o(\hat{m}-m)/m \; \hat{m} = \Omega - \hat{\Omega}$$

and B is a unit vector in the B_o direction. Lastly, the difference quantity R_{ij} is given by*

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* In a more inclusive formulation (really notation) the term (+e) should be replaced by $\hat{e}$, to account for multiply charged ions. Uncer such circumstances R_{ij} would keep the above form with $e/\hat{m}$ replaced by $\hat{e}/\hat{m}$.

(27) $$R_{ij} = \frac{e}{\hat{m}} \hat{p}_{ij} - \frac{e}{m} p_{ij} .$$

We note that in transforming from the two individual momentum equations to the above two equations, (so called, Momentum and Ohm's law (or current) equations respectively), the individual pressures were transformed according to

(28) $$p_{ij}, \hat{p}_{ij} \rightarrow P_{ij}, R_{ij} .$$

This appears to be a natural extension of the transformation set (19) in so far as one of the two new variables, viz., R_{ij}, is of an inherent electric nature. A direct interpretation of R is obtained in the limit where Ohm's law reduces the conservation-like form

(29) $$\frac{\partial}{\partial t} J + \nabla \cdot R_{ij} = 0,$$

so that the time increase of the current density is associated with a negative divergence of R_{ij}, or a net influx of the vector $R_{ij,j}$ (into the related volume element). This is not surprising if we recall that the pressure tensor $\hat{p}_{ij}$ is the rate of transport of i-momentum (of the ions) in the j-direction. If we multiply by $\hat{e}/\hat{m}$ we obtain the rate of transport of i-current (that part due to the ions) in the j-direction. Subtracting $(e/m)p_{ij}$ gives the tensor R_{ij} which is seen to be the rate of transport of the total i-current in the j-direction. We accordingly refer to the R_{ij} tensor as the J-flux tensor.

Finally, we consider the addition of the two energy equations (17, 18) and obtain

$$(30) \qquad \frac{\partial}{\partial t} W + \frac{5}{3} W_o \nabla \cdot U + \nabla \cdot Z = 0.$$

If the total heat flow $Z = \zeta + \hat{\zeta}$ vanishes, then this equation, together with the continuity equation (for ρ), yields the linearized form of the adiabatic law:*

$$(31) \qquad P/P_o = \frac{5}{3} \rho/\rho_o \quad .$$

We have recalled that $\frac{3}{2} P = W = w + \hat{w}$, and that $W_o = 2w_o$.

The physical significance of the electric difference terms is not well defined for these higher moments and we accordingly do not form the counterpart to the above energy equation (30).** Nevertheless, it is interesting to note that the list of new variables, viz., (ρ, Q); (U, J); $(P, R)_{ij}$; suggest the proper form for the relevant difference equation. Noting that the scheme is such that the div. of a successive term is coupled with the time rate of change of a previous variable, e.g., $\dot{\rho} + \rho_o \nabla \cdot U$; $\rho_o \dot{U} + \nabla \cdot P$; $\dot{Q} + \nabla \cdot J$; $\dot{J} + \nabla \cdot R$; and continuing to the energy equation: $1/2 \, (\mathrm{Tr} \, \dot{P}) + \nabla \cdot Z$, it is quite natural to extend the list to include the form, $(1/2) \mathrm{Tr} \, \dot{R} + \nabla \cdot K$, where K is a first order tensor and might appropriately be labelled the R-flux vector.

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* Note that the related sound speed a_o includes the averave mass, $M/2$, viz., $a_o^2 = [\partial P/\partial \rho]_S = \frac{5}{3} [P_o/\rho_o] = \frac{5}{3} (\kappa T_o/\frac{M}{2})$.

** It is formed in Appendix F.

The scheme is physically relevant in as much as each succeeding equation serves to determine a variable which is involved in a previous equation. In this sense once the original set is fixed upon, e.g., (Q,ρ), the proper remaining higher moment variables are defined.*

The reason why we have taken time to formulate certain well-known macroscopic equations is three-fold. Firstly, we have demonstrated that our linearized Boltzmann equations do, indeed, give rise to the standard linearized form of the conservation equations, (i.e., Eqs. 20-23). Secondly, it is revealing to observe the manner in which the stated collision forms appear in the flow equations. Lastly, it is important to have these equations on hand for identification purposes in the analysis section to follow, wherein the formulation is constructed about the distribution function written as functional over the collision terms. The emergent equations are linear combinations of equations which are similar to the above forms. Comparison of these forms enables one to identify the transport coefficients although, in this analysis, we are not specifically interested in this region of investigation, but rather in the characterization of propagation disturbances.

The standard sets of the first three conservation equations, together with Maxwell's equations which give E in terms of

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* H. Grad has suggested changing variables in the Boltzmann equation from $\hat{f}, f$ to F,G: $F = \hat{m}\hat{f} + mf$, $G = \hat{e}\hat{f} - ef$ (Private communication.) (Cf. Appendix F).

U and J, constitute thirteen equations in the 25 unknowns: $(n, u, w, p_{ij}, \zeta; \hat{n}, \hat{u}, \hat{w}, \hat{p}_{ij}, \hat{\zeta}, E)$. Note, $w = (1/2)\mathrm{Tr}\, p_{ij}$. The standard macroscopic technique which is used to close the system involves postulation of the transport coefficients, i.e., phenomenological statements relating higher moments to lower moments. So it is that one defines (for simple gases say) the viscosity η: $P_{ij} = \eta\, \partial U_i/\partial x_i$; and the heat conduction λ: $\zeta_i = \lambda\, \partial T/\partial x_i$. Such statements for the above system would reduce the number of unknowns to thirteen and the system would be determinant. Such phenomenological connection formulas are the essence of the so called macroscopic approach.

A more self-contained technique is that of the microscopic approach wherein one relies only upon the distribution function in order to describe the relevant macroscopic fluctuations in the fluid. Indeed, there are regions of phenomena where certain giant fluctuations in the distribution function itself are observable, (e.g., Larmor resonance, cooperature phenomenon in general). The distinction between macroscopic fluctuations and those just mentioned is that in the latter the resonance regions are seen to be regions about singularities in the distribution function.

In the following section we shall endeavor to employ a microscopic technique in order to uncover the possible modes of development which are excited by the initial perturbation. The starting point will be the said linearized Boltzmann equations.

C. Analysis.

I. Formulation of Homogeneous Equations.

We rewrite the equations (B-II: 1,2) in the more compact notation

$$(1) \qquad \nu g + \frac{\partial g}{\partial t} + \xi \cdot \nabla g - \frac{e}{m}\, \xi \times B_o \cdot \nabla_\xi g = f^{(o)}[h_o + \xi \cdot h_1]$$

$$(2) \qquad \hat{\nu}\hat{g} + \frac{\partial \hat{g}}{\partial t} + \xi \cdot \nabla \hat{g} + \frac{e}{\hat{m}}\, \xi \times B_o \cdot \nabla_\xi \hat{g} = \hat{f}^{(o)}[\hat{h}_o + \xi \cdot \hat{h}_1]$$

where*

$$T_o \hat{\mathcal{T}} = \hat{\nu}\hat{T} - \nu_2 \Delta T$$

$$h_o = \nu \frac{n}{n_o} + \mathcal{T} H_2 , \qquad T_o \mathcal{T} = \nu T + \nu_2 \Delta T$$

$$(3) \qquad h_1 = \nu \frac{u}{C} + \frac{\mathfrak{m}}{m} \nu_1 \frac{\Delta u}{C} - \frac{eE}{mC} , \qquad H_2 = \frac{1}{2}\left[\frac{\xi^2}{C^2} - 3\right]$$

$$\hat{h}_1 = \hat{\nu} \frac{u}{\hat{C}} - \frac{\mathfrak{m}}{\hat{m}} \nu_1 \frac{\Delta u}{\hat{C}} + \frac{eE}{\hat{m}\hat{C}} .$$

In general, to obtain an ion form from an electrom form merely roof all variables and let -e ⟶ +e. We will accordingly work only with the electron equation realizing that at any given point in the analysis the corresponding ion equation could be obtained in this simple manner. The above system is seen to be invariant under the transformation $F \longrightarrow \pm \hat{F}$; $-e \longrightarrow +e$ where F is any variable that is labeled (sign for ν_1, ν_2 terms).

---

* Note change in notation $H^{(2)} \longrightarrow H_2$.

In order to solve for the distribution function in terms of a functional over its moments we will work in the transform k,ω space rather than in the actual x,t space. This is accomplished through the introduction of the Laplace-Fourier transform

$$(4) \qquad X^*(k,\omega) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty \int_{-\infty}^\infty dt\, d^3x\, X(x,t) e^{i\omega t - ik\cdot x} \, ,$$

where X represents all dependent variables which are included in the formulation. The transform of the electron equation appears as:*

$$(5) \qquad [\nu - i\omega + ik\cdot\xi + \Omega \frac{\partial}{\partial\alpha}] g^* = f^{(o)} (h_o^* + \xi\cdot h_1^*) .$$

Let B be a unit vector in the B_o direction. It follows that

$$(6) \qquad \xi_\perp = B\times(\xi\times B) \quad \text{and} \quad k_\perp = B\times(k\times B) .$$

The angle α is such that**

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* Throughout, we tacitly assume the presence of the initial condition terms. The distinction between the resulting dispersion relations is one of words only. In the inhomogeneous case encountered when the initial conditions are explicit, the related roots are associated with singularities of the transform inversion. Those encountered in the homogeneous case insure solubility of the system. The roots are identical; they are both zeros of the master determinant.

** Very simply $\xi\times B_o\cdot\nabla_\xi$ is the projection of ∇_ξ onto $\xi\times B_o$. In cylinder coordinates with ξ_z parallel to B_o, $\partial/\partial\alpha$ is the component of ∇_ξ which is normal to ξ and B_o.

(7) $$k_\perp \cdot \xi_\perp = k_\perp \xi_\perp \cos \alpha .$$

It is the angular cylindrical coordinate of the vector ξ.

The electron Larmor frequency Ω is given by:

(8) $$\Omega = -eB_o/m .$$

The related ion frequency is, $\hat{\Omega} = + eB_o/\hat{m}$.

Dropping the star notation, the equation is further condensed to the form

(9) $$[\bar{a} + ik_\perp \xi_\perp \cos \alpha + \frac{\partial}{\partial \alpha}]g = \frac{\phi}{\Omega} h(\alpha)$$

where g is now the nondimensional distribution measured in units of n_o/C^3 and ϕ is the exponential

(10) $$\phi = \frac{1}{(2\pi)^{3/2}} \exp(- \frac{\xi^2}{2C^2})$$

and

(11) $$\bar{a} = \frac{\nu - i\omega + ik_\parallel \xi_\parallel}{\Omega} ,$$

(12) $$h = h_o + h_1 \cdot \xi \quad ;$$

h is a function of α in as much as it is a function of ξ. Both h and g are periodic in α, so that

(13) $$\begin{aligned} h(\alpha) &= h(\alpha + 2\pi) \\ g(\alpha) &= g(\alpha + 2\pi) . \end{aligned}$$

The integral form of equation (9) appears as:*

$$(14) \qquad g = \frac{(\phi/\Omega)}{e^{4\pi\bar{a}}-1} \int_{0}^{4\pi} \exp\left\{\bar{a}\alpha' \quad \frac{ik_{\perp}\xi_{\perp}}{\Omega} [\sin\alpha - \sin(\alpha+\alpha')]\right\} h(\alpha+\alpha')d\alpha'$$

The closed system of algebraic equations is now obtained by forming the moments, (g and ξ nondimensional),

$$(15) \qquad n = \int g d^3\xi, \quad u = \int g\xi d^3\xi, \quad T = \frac{2}{3} \int g H_2 d^3\xi ,$$

with similar forms for the ion terms.

It is convenient to put the above integral into a more compact form. This is accomplished through introduction of the rotation operator R_θ, which is defined by

$$(16) \qquad R_\theta f(\alpha) = f(\alpha + \theta) .$$

A relevant example follows from the identity

$$(17) \qquad \xi = B(B\cdot\xi) + B\times(\xi\times B) ,$$

so that

$$(18) \qquad \begin{aligned} R_\theta\xi &= B(B\cdot\xi) + \sin\theta(\xi\times B) + \cos\theta\, B\times(\xi\times B) \\ R_{-\theta}\xi &= B(B\cdot\xi) - \sin\theta(\xi\times B) + \cos\theta\, B\times(\xi\times B). \end{aligned}$$

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* This is most simply obtained via the use of the integrating factor: $\exp[\bar{a}\alpha + ik_{\perp}\xi_{\perp} \sin\alpha]$.

Returning to the algebraic formulation we note that any of the above three moments are of the form

$$(19) \qquad \langle H_i \rangle = \int g H_i(\xi) d^3\xi = \int d^3\xi \frac{\phi}{e^{4\pi\bar{a}}-1} \int_0^{4\pi} d\alpha' \exp\left\{\bar{a}\alpha' - \frac{ik_\perp}{\Omega} \cdot [R_{-\pi/2}\xi - R_{\alpha'-\pi/2}\xi]\right\} h(R_{\alpha'}\xi) H_i(\xi) \ .$$

Transforming variables, $\xi \longrightarrow \xi' = R_{+\alpha'/2}\xi$, so that $\xi = R_{-\alpha'/2}\xi'$, we obtain, (dropping the prime on α)

$$(20) \qquad \langle H_i \rangle = \int d^3\xi' \frac{\phi}{2^{4\pi\bar{a}}-1} \int_0^{4\pi} d\alpha \exp\left\{\bar{a}\alpha - \frac{ik_\perp}{\Omega} [R_{-\frac{\pi}{2}-\frac{\alpha}{2}}\xi' - R_{\frac{\alpha}{2}-\frac{\pi}{2}}\xi']\right\} h(R_{\alpha/2}\xi) H_i(R_{-\alpha/2}\xi') \ .$$

If we further set $\psi = \alpha/2$, there results (dropping the prime on ξ)

$$(21) \qquad \langle H_i \rangle = 2 \int d^3\xi \frac{(\phi/\Omega)}{e^{4\pi\bar{a}}-1} \int_0^{2\pi} d\psi \exp\left\{2\bar{a}\psi + \frac{2ik_\perp \cdot \xi_\perp}{\Omega} \sin\psi\right\} h(R_\psi\xi) H_i(R_{-\psi}\xi).$$

At this point an interesting effect of the presence of the steady magnetic field becomes evident. If we recall that ψ is a cylinder coordinate of the vdctor ξ, then the integration involved in the above equation is easily interpreted as being over a four dimensional velocity space. In this sense the magnetic field imposes an extra weighted velocity integration to the three dimensional averaging involved in the formation of moments without an impressed magnetic field. Equations (21) with H_i set equal to 1, ξ, H_2, together with Maxwell's equations constitute a closed set of thirteen algebraic equations in the thirteen unknowns(n, u, T; $\hat{n}$, $\hat{u}$, $\hat{T}$, E).

Symbolically this system may be written

$$(22) \qquad A_{ij}\, X_j = 0 \ .$$

As indicated in the introduction we have chosen to call the 13 x 13 matrix A_{ij}, the master matrix of this problem and the related determinant the master determinant. The relevant modes of excitation are obtained by setting $\det|A_{ij}| = 0$.

The integral coefficients of A_{ij} are functions of (ω, k, $\mathscr{P}$, Ω, ω_o^2, ν, $\hat{\nu}$, ν_1, ν_2, n_o, T_o, κ, m, $\hat{m}$, e), where $\mathscr{P} = ck$, $\omega_o^2 = e^2 n_o/\varepsilon_o \mathcal{M}$; are the only new variables.* These latter two variables enter from the Maxwell equations part of the system, which are written as (all quantities are perturbation quantities, recalling that $Q_o = J_o = 0$),

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* ω_o is written instead of ε_o. Also, the constant μ_o appears through $\mathscr{P} = ck$, since $c^2 = 1/\varepsilon_o\mu_o$. Ω is written for B_o.

$$(23) \qquad \begin{aligned} &\nabla\cdot B = 0 \, , &&\nabla\cdot\varepsilon_o E = Q \\ &\nabla\times E = -\dot{B} , &&\nabla\times B = \mu_o J + \dot{E}/c^2 \, , \end{aligned}$$

c is the speed of light and ε_o and μ_o are the dielectric constant and permeability of free space, respectively.

The latter three equations are combined to give the standard wave equation for $\varepsilon_o E$,

$$(24) \qquad c^2\nabla^2\varepsilon_o E - \varepsilon_o \ddot{E} = \dot{J} + c^2\nabla Q \quad .$$

The transform of this equation is (where variables are now transforms),

$$(25) \qquad \varepsilon_o E = i\left\{\frac{\rho c Q - \omega J}{\omega^2 - \rho^2}\right\} .$$

If this equation is combined with the transform of the continuity equation, viz.,

$$(26) \qquad \omega Q = k\cdot J \ ,$$

one obtains,

$$(27) \qquad \varepsilon_o E = \frac{1}{i\omega}\left\{\frac{\omega^2 J - \rho(\rho\cdot J)}{\omega^2 - \rho^2}\right\} .$$

Recalling that $J = en_o\Delta u$, this latter form may be rewritten as

$$(28) \qquad E = \frac{en_o}{\varepsilon_o i\omega}\left\{\frac{\omega^2\Delta u - \rho(\rho\cdot\Delta u)}{\omega^2 - \rho^2}\right\} \quad .$$

If this expression for E is substituted into the inhomogeneous vector h_1 (cf. Eq. (3)), there results

$$(29) \qquad Ch_1 = \nu u + \nu_3 \Delta u + \frac{\omega_o^2}{i\omega} \left\{ \frac{\wp(\wp \cdot \Delta u)}{\omega^2 - \wp^2} \right\} \quad .$$

The frequency ν_3 is given by

$$(30) \qquad \nu_3 = \frac{\mathscr{m}}{m} \; \nu_1 + \frac{i\omega_o^2 \; \omega}{\omega^2 - \wp^2} \quad .$$

The corresponding ion term is given by:

$$(31) \qquad \hat{\nu}_3 = -\frac{\mathscr{m}}{\hat{m}} \; \nu_1 - \frac{i\hat{\omega}_o^2 \; \omega}{\omega^2 - \wp^2} \quad ,$$

where $\hat{\omega}_o^2 = e^2 n_o / \varepsilon_o \hat{m}$.

In the following section we will introduce $\bar{h}$ which is the non-dimensional form of h, and includes frequencies measured in units of the Larmor frequency Ω; similarly, the frequencies of $\hat{\bar{h}}$ are non-dimensionalized through $\hat{\Omega}$. This process of non-dimensionalization generates a convenient variable $\bar{\nu}_3$ of the two-component plasma which is a function of the reduced mass $\mathscr{m}$:

$$(32) \qquad -\bar{\nu}_3 = (\nu_3/\Omega) = (\hat{\nu}_3/\hat{\Omega}) = -\frac{\nu_1}{\Omega^*} + \frac{i\phi_o^2 \; \omega}{\omega^2 - \wp^2} \quad .$$

Where,

$$(33) \qquad \Omega^* = eB_o/\mathscr{m} ,$$

$$\phi_o^2 = |\omega_o^2/\Omega| = \hat{\omega}_o^2/\hat{\Omega} = en_o/\varepsilon_o B_o \ . \tag{34}$$

The frequency Ω^* is the Larmor frequency of a particle with charge e and mass $\mathcal{M} = m\hat{m}/m+\hat{m}$.

Expressing h_1 in this manner as a function only of $u, \hat{u}$, eliminates E from the problem and reduces the master determinant to a 10×10. Indeed, the homogeneous set $A_{ij}X_j = 0$ is now purely represented by the equation (21), and the related explicit coefficients of A_{ij} are produced by performing the indicated integration over the four-dimensional velocity space, with volume element $d^3\xi \; d\psi$.

C-II. Magnetic Singularities.

Let us recall equation (C-I: 21):

$$I_\ell = \langle H_\ell \rangle = 2 \int d^2\xi \, \frac{\phi}{e^{4\pi\bar{a}}-1} \oint d\psi \, \exp\left\{ 2\bar{a}\psi + 2ik_\perp \cdot \xi_\perp \sin\psi \right\} h(R_\psi \xi) H_\ell(R_{-\psi}\xi) \tag{1}$$

The variable $\bar{h}$ represents the non-dimensional h, and includes frequencies measured in units of the Larmor frequency Ω, and velocities measured in units of the thermal speed C. Similarly, $\hat{h}$ includes frequencies measured in units of the ion Larmor frequency $\hat{\Omega}$, and the ion thermal speed $\hat{C}$. The

$k_\perp \cdot \xi_\perp$ term is non-dimensional and has been written for $[k_\perp \cdot \xi_\perp / \Omega]$. In this term $k_\perp$ is measured in units of the inverse Larmor radius L^{-1}. Note, $L\Omega = C$. Again, $\xi_\perp$ is non-dimensional through the thermal speed C. In general, in the ion equation all explicit speeds are non-dimensionalized through the ion thermal speed and likewise all speeds in the electron equation are non-dimensionalized through the electron thermal speed. A similar statement holds for non-dimensionalizing explicit frequencies through the Larmor frequencies $\hat{\Omega}$ and Ω, respectively.

It is quite clear that in the special case $k_\parallel = 0$ $(\bar{a} \rightarrow a)$ I_ℓ includes regions wherein the related integrations are not well defined. Since these resonances occur about $a = iN$,* i.e., $-i\omega = iN\Omega - \nu$, we see that the related root is of the form, $\omega = \omega_N^{(o)} + \sigma_N k^m + \ldots$. We will show in this section that $m = 2N-2$, for N^{th} order resonance, and derive an explicit form for σ_N.

The variable η_N is defined as:

$$(1') \qquad a = iN + \eta_N \, .$$

We will investigate the limit of $\eta_N \rightarrow 0$, $k_\perp \rightarrow 0$. Knowledge of $\eta_N(k)$ determines the dispersion relation about the N^{th} order electron Larmor frequency. (By N^{th} "order" we mean the N^{th} multiple of Ω.) It should be noted that in the analysis to follow no assumptions are imposed pertaining to the mass ratio, so that the investigation has a complete image formulation about an ion resonance.

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* Actually, they occur about $a = \pm iN$; however, we will restrict the analysis to $a = + iN$, $N > 0$, realizing that an image formulation exists for $a = -iN$.

In the above limit the integral I_ℓ becomes:

$$(2) \qquad I_\ell = \frac{1}{2\pi} \int \phi \, d^3\xi \oint d\psi (\eta_N^{-1} + 2\psi) \exp\left\{ i2N\psi + 2ik_\perp \cdot \xi_\perp \sin\psi \right\} h(R_\psi \xi) H_\ell (R_{-\psi}\xi) \ .$$

Specifically, we are interested in the velocity moment $u = I_1$. We shall see that the reasonance phenomena is characterized by only those velocity moments, owing to a natural splitting of the system (i.e., the master matrix becomes diagonal with a 6 x 6 matrix multiplying the six-component vector $u, \hat{u}$). The most elementary case of such fluctuations would occur in a cold plasma, with a constant-in-time number density, in which all of the particles were rotating in the same direction with say $\omega = \Omega$. In this case, $T = \hat{T} = 0$, $n =$ const., $\hat{n} =$ const., but u is rotating with angular frequency Ω. Note that there is no translation, so that in the usual sense u (or more appropriately J) is not apparently fluctuating. Of course, such variation in J is detected by the electromagnetic radiation emitted.* Such a phenomenon is properly categorized as a giant microscopic fluctuation, even though the formalism suggests a macroscopic variation. Similar arguments apply to pure plasma oscillations. In this sense the two phenomena are related in as much as the plasma oscillations are associated with linear vibrations in J, while magnetic resonances are associated with circular vibrations in J.*

---

* E. G. Harris, personal communication.

The electron macroscopic velocity appears as:

$$(3)\quad u = \frac{1}{2\pi}\int d^3\xi\phi \oint d\psi[\eta_N^{-1} + 2\psi]\exp\left\{2iN\psi + 2iy \sin \psi\right\}$$

$$[h_0 + h_1(B(B\cdot\xi) + (\sin \psi)(\xi \times B) + (\cos\psi)B\times(\xi \times B))][B(B\cdot\xi)$$

$$- (\sin \psi)(\xi \times B) + (\cos \psi)B \times (\xi \times B)] .$$

The variable $y = \xi_\perp \cdot k_\perp$, and $B = B_0/|B_0|$. To evaluate the 2ψ contribution we note that the lowest order non-vanishing terms appear when $k_\perp = 0$. The related integrals are evaluated by rewriting the sins and cos's in their exponential forms, and observing that $\int xe^{ax}dx = [e^{ax}/a^2][xa-1]$. In our case, over the 2π cycle, only the $[xe^{ax}/a]$ term remains. The pertinant integrals are listed below:

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* However, magnetic resonance is clearly not a cooperative phenomenon, but rather an ordered response to an external influence.

$$
\begin{aligned}
&\oint e^{i2N\psi}\,\psi\,\sin\psi\,d\psi = 2\pi/[4N^2-1]\\
&\oint e^{i2N\psi}\,\psi\,\cos\psi\,d\psi = -i4\pi N/[4N^2-1]\\
&\oint e^{i2N\psi}\,\psi\,\sin^2\psi\,d\psi = i\pi/[2N(N^2-1]\\
&\oint e^{i2N\psi}\,\psi\,\cos^2\psi\,d\psi = -i\pi(2N^2-1)/2N(N^2-1]\\
&\oint e^{i2N\psi}\,\psi\,\cos\psi\,\sin\psi\,d\psi = \pi/2(N^2-1]\ .
\end{aligned}
\tag{4}
$$

The first non-vanishing contributions from the η_N^{-1} terms are obtained, again by writing the exponential form for the sin and cos terms and recalling the integral form for J_N: $\oint \exp\left\{iN\psi + iZ\sin\psi\right\}d\psi = 2\pi\,J_N(-Z)$. The pertinent terms are again listed.

$$
\begin{aligned}
&\oint e^{i2N\psi+iZ\sin\psi}\,\sin\psi\,d\psi = -i\pi[J_{2N-1}-J_{2N+1}]\\
&\oint e^{i2N\psi+iZ\sin\psi}\,\cos\psi\,d\psi = -\pi[J_{2N+1}+J_{2N-1}],\\
&\oint e^{i2N\psi+iZ\sin\psi}\,\sin^2\psi\,d\psi = \pi\,J_{2N}-\tfrac{\pi}{2}[J_{2N+2}+J_{2N-2}]\\
&\oint e^{i2N\psi+iZ\sin\psi}\,\cos^2\psi\,d\psi = \pi\,J_{2N}+\tfrac{\pi}{2}\,[J_{2N+2}+J_{2N-2}]\\
&\oint e^{i2N\psi+iZ\sin\psi}\,\sin\psi\,\cos\psi\,d\psi = -\tfrac{i\pi}{2}\,[J_{2N+2}-J_{2N-2}]\ .
\end{aligned}
\tag{5}
$$

The argument of all of these Bessel functions is $Z = 2k_\perp\cdot\xi_\perp$. (Recall k measured in units of L^{-1}, and ξ in units of C.) The expansion of J_N appears as

$$(5) \qquad J_N = \frac{(y/2)^N}{N!} \left\{ 1 + \sum_{m=1}^{\infty} \frac{(-)^m \, N!\,(y/2)^{2m}}{m!\,(N+m)!} \right\} .$$

It follows that as k ⟶ 0, the lowest order terms appear in the quadratic terms

$$(7) \qquad \oint e^{2iN\psi + iy \sin \psi} \left\{ \begin{matrix} \sin^2\psi \\ \cos^2\psi \\ \sin \psi \cos \psi \end{matrix} \right\} d\psi$$

$$= \frac{\pi}{2(2N-2)!} (k_\perp \cdot \xi_\perp)^{2N-2} \left\{ \begin{matrix} -1 \\ +1 \\ +i \end{matrix} \right\} + \ldots$$

while for the 2ψ contribution we repeat

$$(8) \qquad \oint e^{2iN\psi} \ \psi \begin{pmatrix} \sin^2\psi \\ \cos^2\psi \\ \sin \psi \cos \psi \end{pmatrix} d\psi = \frac{i\pi}{2N(N^2-1)} \begin{pmatrix} +1 \\ -(2N^2-1) \\ -iN \end{pmatrix} .$$

Having thus performed the $d\psi$ part of the integration we now form to the $d^3\xi$ part, and immediately note that in this limit the $h_oB(B\cdot\xi)$ term vanishes because the related integrand is odd; while the $h_o\sin\psi$ and $h_o\cos\psi$ terms are discarded because they are higher order in k than are the quadratic terms in $\sin\psi$, $\cos\psi$. In addition, we note the splitting between perpendicular and parallel (to B_o) components. Consider $u_\perp$. The contribution from terms like $h_1\cdot B(B\cdot\xi)$ vanishes because it is odd. Similarly for all of the other cross terms in $\xi_{\shortparallel}\xi_\perp$. At this point it is convenient to introduce the $||F||$ symbol defined by

$$(9) \qquad \lim_{k \to 0} \frac{1}{2\pi} \oint e^{2iN\psi + iy \sin \psi} F(\psi) d\psi = ||F|| .$$

By "lim" we mean either the formal limit if it is non-zero, or if not, then the lowest order term in k.

The $u_\perp$ part of the integral (Eq. 3) becomes in this notation:

$$(10) \qquad u_\perp = \int d^3\xi \; \phi(\xi \cdot B \, \hbar_1)(\xi \times B) \, || - \sin^2\psi(\eta_N^{-1} + 2\psi) ||$$
$$+ (\hbar_1 \cdot \xi_\perp)\xi_\perp \, ||\cos^2\psi(\eta_N^{-1} + 2\psi)|| + [(\xi \cdot B \times \hbar_1)\xi_\perp$$
$$- (\hbar_1 \cdot \xi)\xi \times B] \, ||\sin\psi \cos\psi(\eta_N^{-1} + 2\psi)|| .$$

In general, if V is a vector, then by $V_\perp$ we mean

$$(11) \qquad V_\perp = B \times (V \times B) = V - B(V \cdot B) .$$

For the constant 2ψ contribution we note that if V is independent of ξ, then (cf. Appendix D)

$$(12) \qquad \langle \xi \cdot V \, \xi \rangle = V.$$

The non-constant η_N^{-1} contributions are all of the form, $(V = V_\perp)$

$$(13) \qquad L = \langle (\xi \cdot V)\xi(k_\perp \cdot \xi_\perp)^{2N-2} \rangle$$

This integral is calculated in Appendix D, and is there shown to be

$$L(V) = 2^N N!\, k_\perp^{2N-2}\, \tilde{L}(V) \ . \tag{14}$$

The linear operator $\tilde{L}$ rotates and stretches V according to

$$\tilde{L}(V) = b_{2N}\tilde{k}(\tilde{k}\cdot V) + b_{2N}'\, \tilde{k}\times B(\tilde{k}\times B\cdot V) \ . \tag{15}$$

The unit vector $\tilde{k}$ is in the direction of $k_\perp$, and the b_{2N} coefficients are given by:

$$b_{2N} = \frac{1\cdot 3\cdot 5\cdots(2N-1)}{2\cdot 4\cdots(2N)} \ ; \qquad b_{2N}' = b_{2N}(2N-1)^{-1} \ . \tag{16}$$

There are four distinct contributions to $u_\perp$ of this form. They arise from the terms, $(\xi\cdot B\times\hbar_1)\xi\times B$, $\xi\cdot\hbar_1\xi$, $(\xi\cdot B\times\hbar_1)\xi$, and $(\xi\cdot\hbar_1)\xi\times B$, (noting that $\xi = \xi_\perp = B\times(\xi\times B)$ only, and similarly for $\hbar_1$). These terms, respectively, integrate to, $\tilde{L}(B\times\hbar_1)\times B$, $\tilde{L}(\hbar_1)$, $\tilde{L}(B\times\hbar_1)$, and $\tilde{L}(\hbar_1)\times B$.

Let us rewrite equation (10) as

$$u_\perp = u_\perp^{(o)} + u_\perp^{(1)} \ , \tag{17}$$

where $u_\perp^{(o)}$ is the contribution from the 2ψ integrals, and $u_\perp^{(1)}$ the contribution from the η_N^{-1} integrals. Combining the above yields

$$u_\perp^{(o)} = \frac{1}{2(N^2-1)}\,[-iN\hbar_1 + B\times\hbar_1] \ , \tag{18}$$

$$(19) \qquad u_\perp^{(1)} = \frac{2^{N-2}\, k_\perp^{2N-2}\, N!}{\eta_N (2N-2)!} \Big\{ -B \times \tilde{L}(B \times \hbar_1) + \tilde{L}(\hbar_1) + i[\tilde{L}(B \times \hbar_1) + B \times \tilde{L}(\hbar_1)] \Big\} .$$

From the latter equation we see that in order to maintain a finite phenomenon, η_N must be of the form

$$(20) \qquad \eta_N = k^{2N-2} [\sigma_N + k\sigma_N' + \dots] .$$

The term σ_N is independent of k and is to be determined.

To obtain a closed system of equations, we must include the ion-velocity equation. Since the value of $i\hat{a}$ is not integral about $a = iN$, a more direct means of integration may be employed, which shall be introduced in the succeeding section, the results of which we pre-quote now (cf. Eq. C-III: 34)

$$(21) \qquad \hat{u}_\perp = \frac{1}{\hat{a}^2 + 1} \left\{ \hat{\hbar}_1 \times B + \hat{a}\, \hat{\hbar}_1 \right\} .$$

We recall that ω is such that $a = iN + \eta_N$, so that about this frequency

$$(22) \qquad \begin{aligned} \hat{\Omega}\hat{a} &= (\hat{\nu} - \nu) + iN\Omega + O(k^{2N-2}) \\ \hat{a} &= (\hat{\nu} - \nu)/\hat{\Omega} - iN\, \hat{m}/m + O(k^{2N-2}) . \end{aligned}$$

An immediate eigenfrequency is obtained in the said limit $k \rightarrow 0$, if in addition we let $\wp \rightarrow 0$, and consider the case of small self-collision frequency (ν), but large $\nu_\perp$ (cross-collision frequency). In this limit, (cf. Eq. C-I: 32), we obtain

$$(23) \qquad h_1 = \hat{h}_1 = - \bar{\nu}_3' \Delta u \ ,$$

where $\bar{\nu}_3' = \bar{\nu}_3 \ (\wp = 0)$, and again ω is given by $a = iN + \eta_N$.

Equation (21) for the ion velocity is in the form $u_\perp = \hat{F}\ \Delta u$, and equation (10) for the electron velocity is in the form $u_\perp = F\Delta u$, The F's are the implied linear operators. Subtraction of these two equations yields the closed set $\Delta u = (\hat{F}-F)\Delta u$. The condition for a non-trivial solution yields the desired roots, viz., $\det|\hat{F}-F-I| = 0$. This condition appears as

$$(24) \qquad (z-1)^2 + w^2 = 0$$

where

$$(25) \qquad z = \bar{\nu}_3 \ [\frac{\beta}{\sigma_N} - iNy - \hat{\alpha}\hat{a}]$$

$$(26) \qquad w = \bar{\nu}_3 \ [i \ \frac{\beta}{\sigma_N} + y + \hat{\alpha}] \ .$$

The coefficients, β, y, and α, are

$$(27) \qquad \hat{\alpha} = (\hat{a}^2 + 1)^{-1}, \quad \beta_N = \frac{2^{N-1} N! \ N}{(2N-1)!} \, b_{2N} \ , \quad y = \frac{1}{2} \, (N^2 - 1)^{-1} \ .$$

The solutions to the above secular equation appear as

(28) $$\sigma_N^{(-)} = 0$$

and

(29) $$\sigma_N^{(+)} = 2\beta_N \tau$$

where

(30) $$\frac{1}{\tau} = \frac{i}{2(N-1)} + \frac{1}{\hat{a}-1} = \frac{1}{\overline{\nu_3}} \quad .$$

It follows that there are two roots about each $a = iN$, and these are given by

(31) $$i\omega^{(+)} = \nu - iN\Omega - 2\beta N\Omega\tau\, k_\perp^{2N-2}$$

and

(32) $$i\omega^{(-)} = \nu - iN\Omega - \sigma_N k^{N'} ,$$

where $N' > 2N-2$.*

These relations indicate that the first order decay time is as the inverse self-collision frequency, while to terms of higher order (i.e., terms of $O(k^{2N-2})$), the decay time (i.e., $[\nu - 2\beta_N \Omega \text{ Re } \tau]^{-1}$), is a function of $\hat{a}$ and $\overline{\nu_3}$ through the parameter τ. We immediately note that if

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* That this is so, becomes clear if we write $\tilde{\sigma}_N k^{N'} = k^{2N-2}(\tilde{\sigma}_N k^{N''})$, $N'' > 0$, in which case, to order k^{2N-2}, the "coefficient" $\tilde{\sigma}_N k^{N''}$ would be zero.

$\nu = \nu_1 = 0$ (no dissipation), then Re $\tau = 0$, and there is no decay of these magnetic oscillations. The most significant part of the dispersion is in terms of the frequency $\bar{\nu}_3$, which includes the effect of the number density ($\phi_o = en_o/\varepsilon_o B_o$) and the cross collision frequency ν_1. We note that the explicit orientation of ϕ_o in the formulation is such that the influence of the density of particles on the "pure" resonance $N\Omega$ can be eliminated by a large enough magnetic field, and inversely the gross ordering effect of the steady magnetic field can be weakened by a large enough number density. Since the ratio of plasma frequency to Larmor frequency appears in ϕ_o^2, and is usually quite large, the latter of the two situations would appear to prevail. It should be remembered, however, that these effects are higher order in the expansion parameter k, than are the "pure" resonances. The presence of the other gas (i.e., the ion gas in this case) is most directly exhibited by the presence of the coefficient $\hat{a}$, which is evaluated about the electron resonance, so that

$$(33) \qquad \hat{a} = \hat{m}\left(\frac{\hat{\nu}-\nu}{eB_o} - \frac{iN}{m}\right) + O(k^{2N-2}) \ .$$

An interesting result is obtained if we note the form that a assumes in the inverse calculation about the ion resonances. In this case, a is calculated with ω such that $a = iN + \eta_N$, and appears as

$$(34) \qquad a = m\left(\frac{\hat{\nu}-\nu}{eB_o} - \frac{iN}{\hat{m}}\right) + O(k^{2N-2}) \ .$$

By comparison of these two latter forms it is easily concluded that the effect of the ions on the electron resonances is greater than the effect of the electrons on the ion resonances by a factor which is approximately the mass ratio.

Finally, we note that these resonances are seen to be doublets, with a splitting which is of order k^{2N-2}. This splitting occurs about each resonant frequency, two about the ion frequencies and two about the electron frequencies.*

C-III. Formulation of Generalized Macroscopic Equations.

The standard technique of forming the macroscopic conservation equations by way of moment operation converts a closed system of non-linear integral-differential (Boltzmann) equations into a non-closed system of linear differential (moment) equations. Introduction of the transport coefficients serves to close the latter system and in a phenomenological manner recaptures the effect of the collisions.

The technique of introducing a simplified collision form keeps the effect of the collisions very explicit through the presence of the collision frequencies. Although postulating these collision frequencies is, of course, ad hoc, it is a statement in the microscopic realm, so that the subsequent analysis starts from microscopic origins and is self contained.

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* C. S. Gardner has considered this problem for the case of a collisionless plasma and demonstrates more formally the existence of such double roots, (report to appear).

As has already been noted, the system of equations (C-II: 1) is a closed set of ten homogeneous algebraic equations in (k,ω) space. We will now embark on the formulation of a more explicit form of these equations. Since these operations are on the transformed variables, the related coefficients will be functions of k and ω. Were we to transform back to (x,t) space, a system of integral equations would be obtained which would resemble the integral inverse of the differential macroscopic equations written in Section C-I,* the distinction being that the former set would be a closed system with ten dependent variables. However, as we shall see, a differential system may also be obtained through forming proper linear combinations of the set (C-II: 1).

Still another choice is available if we view the formation of the transform equations in the equally legitimate sense of seeking wave-like solutions of the form $\exp\left(ik\cdot x - i\omega t\right)$ (i.e., as opposed to considering the transform operations as a mathematical artifice which serves only to determine the solution in (x,t) space). This interpretation of seeking plane wave solutions permits the formulation of frequency dependent transport coefficients.

Once again we consider the equations (C-II: 1) and, consistent with the investigation of long wavelength disturbances, set

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* The pseudo inhomogeneous terms are taken to be those arising from the collision forms.

$$(1)\quad e^{2[\bar{a}\psi + iy\sin\psi]} = e^{2\bar{a}\psi}\left\{1 + 2iy\sin\psi - 2y^2\sin^2\psi\right\} + O(k^2)$$

where, as in Section C-II, we recall that $y = k_\perp \cdot \xi_\perp$, and that $\bar{a} = a + ix$, $a = \nu - i\omega$, $x = k_\parallel \cdot \xi_\parallel$ (note, $x + y = k\cdot\xi$).

This present analysis will include terms to $O(k^2)$.

Substitution of the above expansion into Eq. (C-II: 1) yields for the first three moments

$$(2)\quad \frac{n}{2} = \int d^3\xi \, \frac{\phi}{e^{4\pi\bar{a}}-1} \oint d\psi \; e^{2\bar{a}\psi}[1 + 2iy\sin\psi - 2y^2\sin^2\psi]\hbar(R_\psi\xi)$$

$$(3)\quad \frac{3}{3}\,T = \int d^3\xi \, \frac{\phi}{e^{4\pi\bar{a}}-1} \oint d\psi \; e^{2\bar{a}\psi}[1 + 2iy\sin\psi - 2y^2\sin^2\psi]\hbar(R_\psi\xi)\left(\frac{\xi^2-3}{2}\right)$$

$$(4)\quad \frac{u}{2} = \int d^3\xi \, \frac{\phi}{e^{4\pi\bar{a}}-1} \oint d\psi \; e^{2\bar{a}\psi}[1 + 2iy\sin\psi - 2y^2\sin^2\psi]\hbar(R_\psi\xi)R_{-\psi}\xi \;,$$

(with an image set for the ion moments), where we recall that

$$(5)\quad \hbar(R_\psi\xi) = \hbar_o + \hbar_1\cdot\left\{B(B\cdot\xi) + (\sin\psi)\xi\times B + (\cos\psi)B\times(\xi\times B)\right\},$$

$$= \hbar_o + \hbar_1^{\parallel}\xi_\parallel - (\sin\psi)\xi\cdot\hbar_1\times B + (\cos\psi)\xi\cdot\hbar_1^{\perp}$$

$$(6)\quad R_{-\psi}\xi = B(B\cdot\xi) - (\sin\psi)\xi\times B + (\cos\psi)B\times(\xi\times B)\;.$$

Recognizing that all of the $d\psi$ integrations are of a similar form, we define the following functions of $\bar{a} = \frac{\nu - i\omega + ix}{\Omega}$

$$(7) \qquad S^N C^M = \frac{1}{e^{4\pi\bar{a}} - 1} \oint e^{2\bar{a}\psi} \sin^N\psi \cos^N\psi \, d\psi .$$

To perform these integrations, we note the identity

$$(8) \qquad \sin^N\psi \cos^N\psi = \sum_{n=0}^{N+M} d_n \cos n\psi + b_n \sin n\psi \ ,$$

and then write $\sin n\psi$ and $\cos n\psi$ in their equivalent exponential form to obtain

$$(9) \qquad (e^{4\pi\bar{a}} - 1)^{-1} \oint e^{2\bar{a}\psi} \cos n\psi \, d\psi = \mathcal{C}_n = \frac{2\bar{a}}{4\bar{a}^2 + n^2} \ ,$$

$$(10) \qquad (e^{4\pi\bar{a}} - 1)^{-1} \oint e^{2\bar{a}\psi} \sin n\psi \, d\psi = \mathcal{S}_n = \frac{-n}{4\bar{a}^2 + n^2} \ ,$$

so that, for instance,

$$(11) \qquad \begin{aligned} &\mathcal{S}_o = 0 \ , \qquad \mathcal{C}_o = 1/2\bar{a} \\ &n\,\mathcal{C}_n + 2\bar{a}\,\mathcal{S}_n = 0 . \end{aligned}$$

A few of the pertinent integrals are listed below:

(12)

$$c^{0} = \mathcal{C}_{0} \qquad c = \mathcal{C}_{1} \qquad s = \mathcal{S}_{1} \qquad s^{2}-c^{2} = -\mathcal{C}_{2}$$

$$sc = \frac{1}{2}\mathcal{S}_{2} \qquad s^{2} = \frac{1}{4}(\mathcal{C}_{0} - \mathcal{C}_{2}) \qquad c^{2} = \frac{1}{2}(\mathcal{C}_{0} + \mathcal{C}_{2})$$

$$s^{3} = \frac{1}{4}(3\mathcal{S}_{1} - \mathcal{S}_{3}) \qquad c^{3} = \frac{1}{4}(3\mathcal{C}_{1} + \mathcal{C}_{3}) \qquad sc^{2} = \frac{1}{4}(\mathcal{S}_{3} + \mathcal{S}_{1})$$

$$s^{2}c = \frac{1}{4}(\mathcal{C}_{1} - \mathcal{C}_{3}) \qquad s^{4} = \frac{1}{8}(\mathcal{C}_{4} - 4\mathcal{C}_{2} + 3\mathcal{C}_{0}) \qquad c^{4} = \frac{1}{8}(\mathcal{C}_{4} + 4\mathcal{C}_{2} + 3\mathcal{C}_{0})$$

$$s^{3}c = \frac{1}{8}(2\mathcal{S}_{2} - \mathcal{S}_{4}) \qquad s^{2}c^{2} = \frac{1}{8}(\mathcal{C}_{0} - \mathcal{C}_{4}) \qquad c^{3}s = \frac{1}{8}(\mathcal{S}_{4} + 2\mathcal{S}_{2})$$

If these results are substituted into equations (2-4), there remains only the $d^3\xi$ integration. Were we to perform these integrations afortiori with the $S^N C^M$ function of x left intact, the resulting equations would be valid to terms of $O(k_\perp^2)$ for the perpendicular part of the wave number; however, one would obtain a closed formulation for the parallel component $k_\parallel$. This asymmetry, which is inherent in the problem, has already been exhibited in the previous section on magnetic singularities wherein the formulation included terms to $O(k_\perp^{2N-2})$, but only to $O(1)$ in $k_\parallel$.

An example of the opposite extreme is obtained by setting $y = 0$, and examining the resulting formulation with all functions of x left in closed form. The classical Landau damping is included as a special case of this class if we further let all of the dissipative terms vanish (i.e., all collision frequencies equal zero). In this extreme limit the master matrix becomes diagonal so that the number density and parallel component of the electric field are independent of the other variables and are governed by:

$$(13) \qquad \begin{aligned} n &= \frac{-eE_{\parallel}}{mC} \int \frac{d^3\xi \;\phi}{-i\omega + ik_{\parallel}\xi_{\parallel}C} \\ \hat{n} &= \frac{+eE_{\parallel}}{\hat{m}\hat{C}} \int \frac{d^3\xi \;\phi}{-i\omega + ik_{\parallel}\xi_{\parallel}\hat{C}} \end{aligned}$$

(note: k is dimensional).

Poisson's equation is

$$(14) \qquad ik_{\parallel}E_{\parallel} = \frac{en_o}{\varepsilon_o}(\hat{n}-n) \;,$$

and serves to close the system. We recognize these to be the generating equations for the Landau phenomena.[*], [26,27]

The dissipation associated with this collisionless damping is understood to be a diffusion of the charge accumulation regions of the wave. We will be concerned with the more direct dissipation due to collisions.

Returning to the more general formulation we see that the $S^N C^M$ functions are linear combinations of $\mathcal{C}_n$ and $\mathcal{S}_n$. These are all of a similar form and include the factor $[(2\bar{a} + iN)(2\bar{a} - iN)]^{-1}$. Dimensionally these factors appear as $[2\nu - i(2\omega \pm N\Omega) + 2ik_{\parallel}\xi_{\parallel}]^{-1}$. It follows that the $d\xi_{\parallel}$ part of the remaining integration is well defined if[**]

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[*] Cf. footnote p. (42); also to obtain the more standard Laplace form, replace $-i\omega$ by s.

[**] In this manner the magnetic resonances are kept in the formulation. They become explicit after the $d\xi_{\parallel}$ integrations are performed, and appear in the $[S^N C^M]^{(\ell)}$ coefficients.

$\mathrm{Re}\left\{2\nu - 2i\omega \pm iN\Omega\right\} \neq 0$. This condition is assured, for instance, if we restrict the investigation to roots in the domain $\mathrm{Re}(-i\omega) > -\nu$.* Such roots describe phenomena which decay no faster than $e^{-\nu t}$. The expansions now presented are asymptotic to these $d\xi_{\parallel}$ integrations.

If the $d\psi$ integrals in equations (2,3,4) are replaced by the $S^N C^M$ functions (12), there results

$$(15)\quad \frac{n}{2} = \Big\langle C^0\left\{\hbar_0 + \hbar_1^{\parallel}\xi_{\parallel}\right\} - S\left\{\xi\cdot\hbar_1\times B\right\} + C\left\{\xi\cdot\hbar_1^{\perp}\right\} + S\left\{2iy(\hbar_0 + \hbar_1^{\parallel}\xi_{\parallel})\right\}$$

$$- S^2\left\{2iy(\xi\cdot\hbar_1\times B)\right\} + SC\left\{2iy\ \xi\cdot\hbar_1^{\perp}\right\} - S^2\left\{2y^2(\hbar_0 + \hbar_1^{\parallel}\xi_{\parallel})\right\}$$

$$+ S^3\left\{2y^2\xi\cdot\hbar_1\times B\right\} - S^2C\left\{2y^2\xi\cdot\hbar_1^{\perp}\right\} = \langle\mathcal{W}\rangle$$

$$(16)\quad \frac{3}{4}T = \Big\langle\mathcal{W}\Big(\frac{\xi^2 - 3}{2}\Big)\Big\rangle$$

$$(17)\quad \frac{u}{2} = \Big\langle C^0\left\{(\hbar_0 + \hbar^{\parallel}\xi_{\parallel})\xi_{\parallel}\right\} + S\Big\{\xi_{\parallel}[2iy(\hbar_0 + \hbar_1^{\parallel}\xi_{\parallel}) - \xi\cdot h_1\times B]$$

$$- \xi\times B(\hbar_0 + \hbar_1^{\parallel}\xi_{\parallel})\Big\} + C\left\{\xi_{\parallel}\xi\cdot\hbar_1^{\wedge} + \xi_{\perp}(\hbar_0 + \hbar_1^{\parallel}\xi_{\parallel})\right\}$$

$$+ SC\Big\{\xi_{\parallel}(2iy\xi\cdot\hbar_1^{\perp}) - \xi\times B(\xi\cdot\hbar_1^{\perp}) + \xi_{\perp}[2iy(\hbar_0 + \hbar_1^{\parallel}\xi_{\parallel})$$

$$- \xi\cdot\hbar_1\times B]\Big\} + S^2\Big\{-\xi_{\parallel}[2y^2(\hbar_0 + \hbar_1^{\parallel}\xi_{\parallel}) + 2iy\ \xi\cdot\hbar_1\times B]$$

$$+ \xi\times B[\xi\cdot\hbar_1\times B - 2iy(\hbar_0 + \hbar_1^{\parallel}\xi_{\parallel})]\Big\}$$

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* More generally, the integrations are well defined if either $\nu + \mathrm{IM}(\omega) \neq 0$, or $2\mathrm{Re}(\omega) \pm N\Omega + 2x \neq 0$.

$$\begin{aligned}
&+ C^2\{\xi_\perp \xi\cdot\hbar_1^\perp + S^3\xi_\parallel(2y^2\xi\cdot\hbar_1\times B) + \xi\times B[2y^2(\hbar_0+\hbar_1^\parallel\xi_\parallel) \\
&+ 2iy\ \xi\cdot\hbar_1\times B]\} + SC^2\{\xi_\perp 2iy\ \xi\cdot\hbar_1^\perp\} + CS^2\{\xi_\parallel(-2y^2\xi\cdot\hbar_1^\perp) \\
&- \xi_\perp[2y^2(\hbar_0+\hbar_1^\parallel\xi_\parallel) + 2iy\ \xi\cdot\hbar_1\times B] - \xi\times B(\xi\cdot\hbar_1^\perp 2iy)\} \\
&+ S^4\{-\xi\times B\ 2y^2\xi\cdot\hbar_1\times B\} + C^2S^2\{\xi_\perp\xi\cdot\hbar_1^\perp(-2y^2)\} \\
&+ S^3C\{\xi_\perp(2y^2\xi\cdot\hbar_1\times B) + \xi\times B(2y^2\xi\cdot\hbar_1^\perp)\}\rangle \quad .
\end{aligned}$$

Many of these integrals are annihilated immediately by odd-even arguments. The remaining terms appear as:

$$\begin{aligned}
\frac{u}{2} = \langle C^0&\{[\hbar_0+\hbar_1^\parallel\xi_\parallel]\xi_\parallel\} + SC\{\xi_\parallel 2iy\ \xi\cdot\hbar_1^\perp - \xi\times B(\xi\cdot\hbar_1^\perp) \\
&+ \xi_\perp[2iy(\hbar_0+\hbar_1^\parallel\xi_\parallel) - \xi\cdot\hbar_1\times B]\} + S^2\{-2B[\xi_\parallel^2 y^2\hbar_1^\parallel \\
&+ \xi_\parallel iy\ \xi\cdot\hbar_1\times B] + \xi\times B[\xi\cdot h_1\times B - 2iy(\hbar_0+\hbar_1^\parallel\xi_\parallel)]\} \\
&+ C^2\{\xi_\perp\xi\cdot\hbar_1^\perp\} - S^4\{\xi\times B\ 2y^2\xi\cdot\hbar_1\times B\} - C^2S^2\{2y^2\xi_\perp\ \xi\cdot\hbar_1^\perp\} \\
&+ S^3C\{\xi_\perp 2y^2\xi\cdot\hbar_1\times B + \xi\times B\ 2y^2\xi\cdot\hbar_1^\perp\}\rangle
\end{aligned}\tag{18}$$

$$\begin{aligned}
\frac{n}{2} = \langle C^0&\{\hbar_0+\hbar_1^\parallel\xi_\parallel\} - S^2\{2iy\ \xi\cdot\hbar_1\times B\} + SC\{2iy\ \xi\cdot\hbar_1^\perp\} \\
&- S^2\{2y^2\hbar_0\}\rangle
\end{aligned}\tag{19}$$

$$\begin{aligned}
\frac{3T}{4} = \langle\frac{\xi^2-3}{2}&[C^0\{\hbar_0+\hbar_1^\parallel\xi_\parallel\} - S^2\{2iy\ \xi\cdot\hbar_1\times B\} \\
&+ SC\{2iy\ \xi\cdot\hbar_1^\perp\} - S^2\{2y^2\hbar_0\}]\rangle \quad .
\end{aligned}\tag{20}$$

The $S^N C^M$ functions of x are expanded as

$$(21) \qquad S^N C^M = [C^N C^M]^{(0)} - ix[S^N C^M]^{(1)} - x^2 [S^N C^M]^{(2)} + \dots ,$$

which serves to define the coefficients $[S^N C^M]^{(\ell)}$.

Recalling the definition of the $S^N C^M$ functions (cf. Eq. 7), we see that the relevant coefficients involve the following four forms:

$$(22) \qquad \begin{aligned}
(\bar{a}^2 + 1)^{-1} &= (a^2 + 1)^{-1} - ix \frac{2a}{(a^2+1)^2} - x^2 \frac{(3a^2-1)}{(a^2+1)^3} + \dots \\
\frac{\bar{a}}{\bar{a}^2 + 1} &= \frac{a}{a^2 + 1} - ix \frac{(a^2 - 1)}{(a^2+1)^2} - x^2 \frac{a(a^2-3)}{(1+a^2)^3} + \dots \\
\frac{1}{\bar{a}(\bar{a}^2+1)} &= \frac{1}{a(a^2+1)} - ix \frac{(3a^2 + 1)}{a^2(a^2+1)^2} - x^2 \frac{6a^4 + 3a^2 + 1}{a^3(a^2 + 1)^3} + \dots \\
(2\bar{a})^{-1} &= (2a)^{-1} - ix(2a^2)^{-1} - x^2(2a^3)^{-1} + \dots
\end{aligned}$$

The related coefficients appear as:

$$(23) \qquad \begin{aligned}
&[C^o]^{(0)} = (2a)^{-1}, \quad [S^2]^{(o)} = [4a(a^2+1)]^{-1}, \quad [C^2]^{(o)} = \frac{1}{4a}\left(\frac{2a^2+1}{a^2+1}\right), \\
&[SC]^{(o)} = -[4(a^2+1)]^{-1}, \quad [C^2 - S^2]^{(o)} = a[2(a^2+1)]^{-1}, \\
&[S^3C]^{(o)} = -\tfrac{3}{8}[(a^2+4)(a^2+1)]^{-1}, \quad [S^2C^2]^{(o)} = [16a(a^2 + 1)]^{-1} \\
&[S^4]^{(o)} = 3[4a(a^2+1)(a^2+4)]^{-1}, \quad [SC]^{(1)} = \frac{-a}{2(1+a^2)^2},
\end{aligned}$$

(23)

$$[S^2]^{(1)} = \frac{a}{(1+a^2)^2}\,, \quad [C^0]^{(2)} = (2a^3)^{-1}\,, \quad [C^2-S^2]^{(2)} = \frac{a(a^2-3)}{2(1+a^2)^3}\,,$$

$$[S^2]^{(2)} = \frac{6a^4+3a^2+1}{4a^3(a^2+1)^3}$$

$$[SC]^{(2)} = -\frac{1}{4}\left(\frac{3a^2+1}{(a^2+1)^3}\right) .$$

III-a. The energy* equations.

Substituting these forms into the above equations, the $\langle\ \rangle$ integrals (i.e., $d^3\xi$) are readily performed (cf. Appendix D). Let us first consider the n and T equations:

$$(24) \quad \frac{n}{2} = \nu n[C^0]^{(0)} - k_{\parallel}^2[C^0]^{(2)}(\nu n + \mathcal{T}) + 2i[S^2]^{(0)}\hbar_1\ k\times B$$
$$+ 2i[SC]^{(0)}k_{\perp}\cdot\hbar_1 - 2k_{\perp}^2[S^2]^{(0)}(\nu n + \mathcal{T}) - i[C^0]^{(1)}k_{\parallel}\hbar_1''$$

$$(25) \quad \frac{3T}{4} = \frac{3}{2}\mathcal{T}\,[C^0]^{(0)} - k^2[C^0]^{(2)}(\nu n + \tfrac{7}{2}\mathcal{T})$$
$$+ 2i[S^2]^{(0)}\hbar_1\cdot k\times B + 2i[SC]^{(0)}k_{\perp}\cdot\hbar_1$$
$$- 2k^2[S^2]^{(0)}(\nu n + \tfrac{7}{2}\mathcal{T}) - 2i[C^0]^{(1)}k_{\parallel}\ \hbar_1''$$

where we have set

$$(26) \quad \begin{aligned} \mathcal{T} &\equiv \nu T + \nu_2\Delta T = (\nu-\nu_2)T + \nu_2\hat{T} \\ \hat{\mathcal{T}} &\equiv \hat{\nu}\hat{T} - \nu_2\Delta T = (\hat{\nu}-\nu_2)\hat{T} + \nu_2 T\ . \end{aligned}$$

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[*] Cf. footnote, p. (110).

Substraction of the temperature equation from the number density equation yields, (returning to dimensional form)

$$(27) \qquad \frac{\partial}{\partial t}\left\{\frac{p}{p_o} - \frac{5}{3}\frac{n}{n_o}\right\} = \nu_2\frac{\Delta T}{T_o} + \frac{5}{3}c^2\left\{\frac{\nabla_\perp^{\,2}}{a^2+\Omega^2} + \frac{\nabla_{\shortparallel}^{\,2}}{a^2}\right\}\frac{\mathcal{T}}{T_o}$$

$$(27.1) \qquad \frac{\partial}{\partial T}\left\{\frac{\hat{p}}{p_o} - \frac{5}{3}\frac{\hat{n}}{n_o}\right\} = -\nu_2\frac{\Delta T}{T_o} + \frac{5}{3}\hat{c}^2\left\{\frac{\nabla_\perp^{\,2}}{\hat{a}^2+\hat{\Omega}^2} + \frac{\nabla_{\shortparallel}^{\,2}}{\hat{a}^2}\right\}\frac{\hat{\mathcal{T}}}{T_o}$$

Again we have partially inverted back to (x,t) space, which enables us to obtain in this instance an ω-dependent heat conductivity.

If this equation is compared with the eliminant of the individual energy equation (B-II: 16) and the continuity equation (B-II: 7), viz.,

$$(28) \qquad \frac{\partial}{\partial t}\left(\frac{p}{p_o} - \frac{5}{3}\frac{n}{n_o}\right) + \frac{\nabla\cdot\zeta'}{w_o} = \nu_2\frac{\Delta T}{T_o}$$

one obtains the following expressions for the components of the individual electron heat conductivities:

$$(29) \qquad \begin{aligned} \zeta_\perp &= -\frac{5}{3}\frac{w_o c^2}{a^2+\Omega^2}\nabla_\perp\left\{\frac{\nu T+\nu_2\Delta T}{T_o}\right\} - \frac{\hat{m}}{\hat{m}+m}w_o\Delta u_\perp \\ \zeta_{\shortparallel} &= -\frac{5}{3}\frac{w_o c^2}{a^2}\nabla_{\shortparallel}\left\{\frac{\nu T+\nu_2\Delta T}{T_o}\right\} - \frac{\hat{m}}{\hat{m}+m}w_o\Delta u_{\shortparallel}\ . \end{aligned}$$

Similarly for the individual ion heat conductivities we find

$$\hat{\zeta}_\perp = -\frac{5}{3}\frac{w_0\hat{c}^2}{\hat{a}^2+\hat{\Omega}^2}\nabla_\perp\left\{\frac{\hat{\nu}\hat{T} - \nu_2\Delta T}{T_0}\right\} + \frac{m}{\hat{m}+m}w_0\Delta u_\perp$$

(30)

$$\hat{\zeta}_\parallel = -\frac{5}{3}\frac{w_0 c^2}{\hat{a}^2}\nabla_\parallel\left\{\frac{\hat{\nu}\hat{T} - \nu_2\Delta T}{T_0}\right\} + \frac{m}{\hat{m}+m}w_0\Delta u_\parallel$$

The more relevant total heat flow is merely the addition of the above forms and is given by:

$$Z_\parallel = -\frac{5}{3}\frac{w_0}{T_0}\nabla_\parallel\left\{\frac{\hat{c}^2}{\hat{a}^2}(\hat{\nu}\hat{T} - \nu_2\Delta T) + \frac{c^2}{a^2}(\nu T + \nu_2\Delta T)\right\} - \left(\frac{\hat{m}-m}{\hat{m}+m}\right)w_0\Delta u_\parallel$$

(31)

$$Z_\perp = -\frac{5}{3}\frac{w_0}{T_0}\nabla_\perp\left\{\frac{\hat{c}^2}{\hat{a}^2+\hat{\Omega}^2}(\hat{\nu}\hat{T} - \nu_2\Delta T) + \frac{c^2}{a^2+\Omega^2}(\nu T + \nu_2\Delta T)\right\} - \left(\frac{\hat{m}-m}{\hat{m}+m}\right)w_0\Delta u_\perp .$$

In the limit of $m = \hat{m}$, $T = \hat{T}$ and $\nu \gg \omega$, the above yields the Chapman-Cowling result,(18)

(32) $$\frac{Z_\perp}{Z_\parallel} = \frac{1}{1+(\Omega/\nu)^2} .$$

More generally, if $\Delta T \neq 0$, we see that the heat flow is proportional to the gradient of a linear combination of both temperatures. In addition the contributions from the difference in flow terms indicates that, even to the lowest order, the gas mixture appears non-adiabatic.(18,19)

III-b. The continuity equations.

To obtain the continuity equations we form a linear combination of the number density equation (24) and the velocity equation. They appear as:

$$(33) \qquad \frac{n}{2} - \nu n[C^0]^{(0)} = -k_{\parallel}^2[C^0]^{(2)}(\nu n + \mathcal{T}) + 2i[S^2]^{(0)}\hbar_1 \cdot k \times B$$

$$+ 2i[SC]^{(0)}k \cdot \hbar_1 - 2k^2[S^2]^{(0)}(\nu n + \mathcal{T}) - i[C^0]^{(1)}k_{\parallel}\hbar_1^{\parallel} \;.$$

$$(34) \qquad \frac{u}{2} = - ik_{\parallel}[C^0]^{(1)}(\nu n + \mathcal{T}) + \hbar_1^{\parallel}[C^0]^{(0)} - 3k_{\parallel}^2\hbar_1^{\parallel}[C^0]^{(2)}$$

$$+ 2[SC]^{(1)}k_{\parallel}k_{\perp}\cdot\hbar_1 - 2h_1 \times B[SC]^{(0)} + 2h_1 \times B\; k_{\parallel}^2[SC]^{(2)}$$

$$+ 2ik[SC]^{(0)}(\nu n + \mathcal{T}) + 2\hbar_1^{\parallel}[SC]^{(1)}k_{\parallel}k_{\perp} - 2k_{\perp}^2\hbar_1^{\parallel}[S^2]^{(0)}$$

$$+ 2[S^2]^{(1)}k_{\parallel}\hbar_1 \cdot k \times B + \hbar_1^{\perp}[C^2-S^2]^{(0)} - k_{\parallel}^2\hbar_1[C^2-S^2]^{(2)}$$

$$- 2[S^4]^{(0)}\left\{k_{\perp}^2\hbar_1 + 2(k \times B)\hbar_1 \cdot k \times B\right\}$$

$$- 2[C^2S^2]^{(0)}\left\{k_{\perp}^2 h_1 + 2k_{\perp}k_{\perp}\cdot h_1\right\} + 2[S^3C]^{(0)}\left\{-2k_{\perp}^2 B \times h_1\right.$$

$$\left. - 2k_{\perp}h_1 \cdot k \times B + 2k \times B\; h_1 \cdot k_{\perp}\right\} \;.$$

The linear combination is $1/2(\nu n + ik\cdot u)$, and to $O(k^2)$ is

$$(35) \qquad \frac{1}{2}\,(an + ik\cdot u) - \nu na[C^0]^{(0)} = k_{\parallel}^2(\nu n + \mathcal{T})\left\{[C^0]^{(1)} - a[C^0]^{(2)}\right\}$$

$$- ik_{\parallel}\hbar_1^{\parallel}\left\{-a[C^0]^{(1)} + [C^0]^{(0)}\right\} - 2[ik\cdot\hbar_1 \times B$$

$$+ k^2(\nu n + \mathcal{T})]\Big\{a[S^2]^{(o)} + SC^{(o)}\Big\} + ik\cdot h_1^{\perp}\Big\{ 2a[SC]^{(o)}$$

$$+ [C^2 - S^2]^{(o)}\Big\} + O(k^3) \ .$$

Reference to the table of functions on p. 69 , indicates that all of the terms on the right hand side vanish identically, so that to $O(k^2)$ we obtain the standard continuity equations:

$$\begin{aligned} &\omega n - n_o k\cdot u = 0 \\ (36) \qquad & \\ &\omega\hat{n} - n_o k\cdot\hat{u} = 0 \ . \end{aligned}$$

It should be noted in passing that although these equations are here obtained to $O(k^2)$, they were also derived in Section B-II through the more standard procedure of moment constructions, from which it follows that they are valid to all orders in k.

III-c. The momentum equations.

To invert the velocity equations we change from a vector representation to a cartesian representation which is defined with respect to the left-hand triplet of unit vectors $[\tilde{k}_\perp, \tilde{k}_\perp \times B, B]$. The system is such that if V is a vector then $V_z = B\cdot V$, $V_x = \tilde{k}\cdot V$, $V_y = \tilde{k}\times B\cdot V$. In forming the three components of the velocity equation we note the following identities, (calling $\tilde{k}_\perp$, $\tilde{k}$)

$$\tilde{k}\cdot h_1 \times B = -h_1\cdot\tilde{k}\times B = -h_1^y$$

$$\tilde{k}\cdot h_1 = h_1^x$$

$$\tilde{k}\times B\cdot h_1\times B = h_1\cdot B\times(\tilde{k}\times B) = h_1\cdot\tilde{k} = h_1^x \tag{37}$$

$$\tilde{k}\times B\cdot h_1 = h_1^y$$

$$\tilde{k}\cdot k = k_x \;, \quad \tilde{k}\times B\cdot k\times B = k_x \;, \quad k_y = 0 \qquad .$$

The three components of equation (C-III: 34) appear as

$$\begin{aligned} \frac{u_z}{2} = {} & -ik_z[C^o]^{(1)}(\nu n + \mathcal{T}) + \hbar_1^z\Big\{[C^o]^{(0)} - 3k_z^2[C^o]^{(2)} \\ & - 2k^2[S^2]^{(0)}\Big\} + \hbar_1^x\Big\{2k_zk_x[SC]^{(1)}\Big\} + \hbar_1^y\Big\{2k_zk_x[S^2]^{(1)}\Big\} \end{aligned} \tag{38}$$

$$\begin{aligned} \frac{u_x}{2} = {} & 2ik_x[SC]^{(0)}(\nu n + \mathcal{T}) \quad \hbar_1^z\Big\{2k_xk_z[SC]^{(1)}\Big\} \\ & + \hbar_1^x\Big\{[C^2-S^2]^{(0)} - k_z^2[C^2-S^2]^{(2)} - 2k_x^2([S^4]^{(0)} \\ & + S[C^2S^2]^{(0)})\Big\} + \hbar_1^y\Big\{2[SC]^{(0)} - 2k_z^2[SC]^{(2)} - 8k_x^2[S^3C]^{(0)}\Big\} \end{aligned} \tag{39}$$

$$\begin{aligned} \frac{u_y}{2} = {} & -2ik_x[S^2]^{(0)}(\nu n + \mathcal{T}) + \hbar_1^z\Big\{-2k_xk_z[S^2]^{(1)} \\ & + \hbar_1^x\Big\{-[SC]^{(0)} + 2k_z^2[SC]^{(2)} + 8k_x^2[S^3C]^{(0)}\Big\} \\ & + \hbar_1^y\Big\{[C^2-S^2]^{(c)} - k_z^2[C^2-S^2]^{(2)} - 2k_x^2(3[S^4]^{(0)} + [C^2S^2]^{(0)})\Big\}. \end{aligned} \tag{40}$$

If the u_x equation is multiplied by a and then added to the u_y equation, there results:

$$(41) \qquad +u_y + au_x + \frac{ik_x}{a}(\nu n + \mathcal{Z}) - \hbar_1^x + \mathcal{A}_{x\ell}\hbar_1^\ell = 0$$

The $\mathcal{A}_{\ell j}$ matrix is a function of the $[C^n S^n]^{(n)}$ coefficients.

In similar manner if the u_y equation is multiplied by $-a$, and then added to the u_x equation, we obtain

$$(42) \qquad -u_x + au_y + \hbar_1^y + \mathcal{A}_{y\ell}\hbar_1^\ell = 0 \quad .$$

Finally, the z equation is rewritten in the similar manner:

$$(43) \qquad au_z + \frac{ik_z}{a}(n + \mathcal{T}) - \hbar_1^z + \mathcal{A}_{z\ell}\hbar_1^\ell = 0 \ .$$

These three equations are the components of the more familiar vector equation (return to dimensional notation by multiplying by ΩC; recall that $\Omega = eB/m$), where we have recalled that $(\tilde{x}, \tilde{y}, \tilde{z})$ is a left-hand triplet and that $k = (k_x, 0, k_z)$:

$$(44) \qquad n_o \frac{\partial u}{\partial t} + \frac{en_o}{m}(E + u \times B) + \frac{\nu}{a\ m}\nabla p + \mathcal{A}_{ij} \cdot h_1$$

$$= \nu_1 \frac{\mathfrak{m}}{m} n_o \Delta u - \frac{\nu_2}{a}\frac{n_o \kappa}{m}\nabla(\Delta T) \ .$$

Similarly, for the ion equation we have

$$(45) \qquad n_o \frac{\partial\hat{u}}{\partial t} - \frac{en_o}{\hat{m}}(E + \hat{u} \times B) + \frac{\hat{\nu}}{\hat{a}\hat{m}} \nabla\hat{p} + \mathcal{A}_{ij}\cdot\hat{h}_1$$

$$= -\nu_1 \frac{\mathfrak{m}}{m} n_o \Delta u + \frac{\nu_2}{\hat{a}} \frac{n_o}{\hat{m}} \nabla(\Delta T) .$$

We have written p for $\kappa(n_o T + T_o n)$, cf. Eq. (B-I: 5).

The a^{-1} operator can be interpreted in either of two ways. First, let us call $\nu a^{-1}p$ the augmented pressure $\underset{|}{p}$, so that (recall that these transformed variables have carried an unwritten star -- cf. p.142),

$$(46) \qquad \nu a^{-1} p^* = \underset{|}{p}^*$$

The inverse of this equation appears as

$$(47) \qquad \underset{|}{p} = \underset{|}{p}(0)e^{-\nu t} + \nu \int_0^t e^{-\nu(t-t')} p(t')dt'$$

or

$$(48) \qquad \underset{|}{p} = p + e^{-\nu t}[\underset{|}{p}(0) - p(0)] + \int_0^t e^{-\nu(t-t')} p_{t'}(t')dt' .$$

The first term is the standard pressure, while the second indicates the influence of the initial pressure, which is seen to be very small after only a few collisions. The last term is a non-local effect over past pressure which, for large collision frequencies, becomes almost local.*

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* It can be seen immediately from Eq. (46) that as $\nu \to \infty$, $\underset{|}{p} \to p$.

The second interpretation comes about when we note the identity

$$(49) \qquad \nu(\nu - i\omega)^{-1} = \left\{ 1 + (i\omega/\nu) + \ldots + (i\omega/\nu)^{n-1} \right\} + \frac{\nu(i\omega/\nu)^n}{\nu - i\omega} \quad .$$

This expression serves to change the integral operator to a series of differential operations with a high order integro-differential remainder.

In the limit of finite wave speeds (MHD), to be discussed momentarily, $(\omega/k) \rightarrow \text{const}$. Since the gradient operator already multiplies the augmented pressure, we see that in this limit

$$(50) \qquad \nabla \text{ꝑ} = \nabla p + O(k^2) \quad .$$

It follows that in first order theory the pressure term becomes the standard kinetic pressure term with $p = \kappa(nT_o + Tn_o)$.

Referring to equation (27), we see that $\Delta T = O(k)$. Since there is a gradient operating on this inhomogeneous term in our momentum equations, it follows that the related term is of $O(k^2)$.

We note that to $O(k)$, h_1 has been eliminated in the above equations (44,45). This was done in order to recapture the more familiar form of the momentum equations, although by the analysis of Section B-II, we already knew that these equations were contained in the model. Since the forms with h_1 kept explicit and those with h_1 eliminated (in favor of

E, $\hat{u}$, u) are related by linear combinations, the resultant dispersion relations of the two schemes are identical. With this point in mind the above set (27; 36; 44,45) is taken as our final form of the velocity moment equations to $O(k^2)$.

We now list the elements of $\mathcal{A}_{ij}$ to note the explicit dependence on $\cos\theta = B\cdot k/Bk$ and for reference in higher order calculations. In addition, if we compare the momentum equations of this section with equations (B-II: 12,13), we see that the elements of $\mathcal{A}_{ij}$ enter inthe identification of the stress and electrical conductivity.*

$$(51)\qquad \frac{\mathcal{A}}{\nu k^2} = \begin{pmatrix} \left\{k_z^2\mathcal{L} + k_x^2\mathcal{L}'\right\} & \left\{2k_xk_z\mathcal{D}\right\} \\ & \\ \left\{2k_xk_z\mathcal{F}\right\} & \left\{k_z^2\mathcal{B} + k_x^2\mathcal{B}'\right\} \end{pmatrix} = \left(\begin{array}{c|c} 2\times 2 & 2\times 1 \\ \hline 1\times 2 & 1\times 1 \end{array}\right)$$

The orientation is in the order x,y,z, with B in the z direction and k in the x direction. The submatrices are given by

$$(52)\qquad \frac{\mathcal{L}}{2} = \begin{pmatrix} \left\{2[SC]^{(2)} - a[C^2-S^2]^{(2)}\right\} & -\left\{2a[SC]^{(2)}+[C^2-S^2]^{(2)}\right\} \\ -\left\{2a[SC]^{(2)} + [C^2-S^2]^{(2)}\right\} & -\left\{2[SC]^{(2)}-a[C^2-S^2]^{(2)}\right\} \end{pmatrix}$$

---

* Recall that h_1 is composed of three parts. The E and Δu terms contribute to the conductivity, while the remaining u (and Δu) terms contribute to the viscosity.

$$\frac{\mathcal{L}'}{2} = \begin{pmatrix} -\left\{8[s^3c]^{(0)} - 6a[s^2c^2]^{(0)} - 2a[s^4]^{(0)}\right\} & +\left\{8[s^3c]^{(0)} + [s^4]^{(0)} + 2[c^2s^2]^{(0)}\right\} \\ -\left\{8a[s^3c]^{(0)} + 2[s^4]^{(0)} + 6[s^2c^2]^{(0)}\right\} & -\left\{8[s^3c]^{(0)} - 6a[s^4]^{(0)} - 2a[c^2s^2]^{(0)}\right\} \end{pmatrix}$$

$$\frac{\mathcal{D}}{2} = \begin{pmatrix} \left\{a[sc]^{(1)} + [s^2]^{(1)}\right\} \\ \left\{[sc]^{(1)} + a[s^2]^{(1)}\right\} \end{pmatrix} , \quad \frac{\mathcal{F}}{2a} = \begin{pmatrix} \left\{[sc]^{(1)}\right\} & \left\{[s^2]^{(1)}\right\} \end{pmatrix}$$

$$\frac{\mathcal{B}}{2a} = \left\{3[c^0]^{(2)}\right\} , \quad \frac{\mathcal{B}'}{2a} = \left\{2[s^2]^{(0)}\right\} .$$

The explicit dependence on a of all of these coefficients appears on p. 69.

The aggregate of equations which we have called the energy, continuity, and momentum equations are seen to form a closed set of linear equations in the ten unknowns (n, u, T; $\hat{n}$, $\hat{u}$, $\hat{T}$). These equations are formulated to $O(k^2)$; however, their ω-dependence is in closed form.

In the succeeding section we will consider the various roots of the related Master determinant. We shall see how the spectrum of plasma phenomena is generated by examining the different forms of solutions, ω = ω(k).

C-IV. Dispersion Relations.

It is convenient at this point to list the system of equations derived in the previous section which, together with Maxwell's equations, form a closed aggregate:

(1) $$\frac{\partial n}{\partial t} + n_o \nabla \cdot u = 0$$

(2) $$\frac{\partial \hat{n}}{\partial t} + n_o \nabla \cdot \hat{u} = 0$$

(3) $$\frac{\partial}{\partial t}\left\{\frac{p}{p_o} - \frac{5}{3}\frac{n}{n_o}\right\} = \nu_2 \frac{\Delta T}{T_o} + \frac{5}{3} c^2 \left\{\frac{\nabla_\perp^2}{a^2 + \Omega^2} + \frac{\nabla_{\shortparallel}^2}{a^2}\right\} \frac{\mathcal{T}}{T_o}$$

(4) $$\frac{\partial}{\partial t}\left\{\frac{\hat{p}}{p_o} - \frac{5}{3}\frac{\hat{n}}{n_o}\right\} = -\nu_2 \frac{\Delta T}{T_o} + \frac{5}{3} \hat{c}^2 \left\{\frac{\nabla_\perp^2}{\hat{a}^2 + \hat{\Omega}^2} + \frac{\nabla_{\shortparallel}^2}{\hat{a}^2}\right\} \frac{\hat{\mathcal{T}}}{T_o}$$

(5) $$n_o \frac{\partial u}{\partial t} + \frac{e n_o}{m}(E + u \times B_o) + \frac{\nu}{am}\nabla p + \mathcal{A}_{ij} \cdot h_1$$
$$= \nu_1 \frac{\mathfrak{m}}{m} n_o \Delta u - \frac{\nu_2 n_o \kappa}{am}\nabla(\Delta T)$$

(6) $$n_o \frac{\partial \hat{u}}{\partial t} - \frac{e n_o}{\hat{m}}(E + \hat{u} \times B_o) + \frac{\hat{\nu}}{\hat{a}\hat{m}}\nabla \hat{p} + \hat{\mathcal{A}}_{ij} \cdot \hat{h}_1$$
$$= -\nu_1 \frac{\mathfrak{m}}{\hat{m}} n_o \Delta u + \frac{\nu_2 n_o \kappa}{\hat{a}\hat{m}}\nabla(\Delta T)$$

(7) $$\nabla \times B = \mu_o J = \dot{E}/c^2$$

(8) $$\nabla \times E = -\dot{B}$$

(9) $$\nabla \cdot E = Q/\varepsilon_o$$

(10) $\nabla \cdot B = 0$.

The presence of the a terms symbolically represent integral operators over past time.* The essence of these operators is such that for large collision frequencies (i.e., large compared to the disturbance frequency), they become extremely local. This is seen most directly from the coefficients themselves, since $a = \nu - i\omega$. An equivalent statement which follows from examining the related integral inversion is that the terms become local after a sufficient number of collision times.

Let us consider the order in ω of the transform of the above system in the limit $(\hat{\nu}, \nu \gg \omega)$, so that all coefficients are local in time. This set is seen to be a system of equations for the variables $(\hat{n}, n, \hat{u}, u, \hat{T}, T, B, E)$. Each of the first eight equations is a first order differential equation in time for each of the above eight variables, respectively. The remaining two equations (9, 10) serve to eliminate one component of E and B respectively in terms of other variables of the system, so that only two of the three components of the vector equation (7) and two of the three components of the vector equation (8) are relevant. The remaining two components are redundant. If we envision the related coefficient determinant of the transformed system, the fourteen diagonal elements each contain an ω entry. It follows that the related dispersion relation is 14^{th} order in ω. Oster[25] considers a system of equations similar to the above set, except that they are temperature independent, and do not include the effect of collisions. The Maxwell's equations part of this system excludes the two divergence equations. The related determinant appears 16^{th} order in ω.

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* Symbolically, in the sense that the presence of the inversion integral is tacitly assumed.

But we have already seen that such a system should yield a 14^{th} order relation. In addition, the related system implies that the time derivatives of the divergence equations are, respectively, zero. Since the divergence equations are not included in this system, two of the fourteen roots must be zero, so that, in all, four of Oster's sixteen roots should degenerate to zero.

Returning to the above system (1-10), if terms of $O(k)$ are retained, but the restrictions on the self-collision frequencies $(\nu, \hat{\nu})$ are removed, then specifically, the transformed momentum equations now include an ω factor which multiplies u, and $\hat{u}$, respectively. It follows that in this more general case the dispersion relation is 16th order in ω.

The higher order $\mathcal{P}_{ij}$ and ΔT terms of equations (3-6) can be seen to contribute to the stress by comparison with equations (B-II: 12,13). However, these terms are not in standard form*, and are more useful in the investigation of higher order dispersion relations.

We will concern ourselves in this section with the lowest order, non-trivial dispersion relations which emerge from these equations.

By transforming back to (k,ω) space an algebraic, homogeneous set of ten equations in the said ten unknowns is obtained. The condition that a non-trivial solution exists (viz., that the master determinant vanish) yields the desired relations, which may be written, $\omega = \omega(k)$.

Consistent with the expansions involved in deriving the above set of equations, we similarly expand the dispersion relation about small k (cf. Appendix B),

$$(11) \qquad \omega = \omega^{(o)} + \tilde{k}\omega^{(1)} + \tilde{k}^2\omega^{(2)}$$

($\tilde{k}$ is non-dimensional).

The speed of light is included in the following manner. It is either taken to be constant with respect to the limiting process $k \rightarrow 0$, or it is taken to be $O(k)^{-1}$ in this limiting process. In the latter case, the parameter $\wp = ck$ is constant as $k \rightarrow 0$. Since this parameter $\wp$ enters in the Maxwell's equations part of the system, which have been kept intact, it follows that the resulting formulation is closed in the variable $\wp$.

---

* By "standard" we mean a form from which the viscosity coefficient may readily be identified.

The series (11) yields two distinct classes of roots, according to which $\omega^{(o)} = 0$, or $\omega^{(o)} \neq 0$. The distinction between these classes is physically relevant. The first categorization is indicative of finite wave speeds $[\lim_{k \to o} (\omega/k) = \omega^{(1)}]$. The latter is characteristic of phenomena associated with infinite phase speeds, since in this instance $\lim_{k \to o} (\omega/k)$ is unbounded. Let us consider the parameter c with respect to both these limits. In the first instance (finite wave speeds) the ratio $(\omega/k)/c$ is zero if $c = O(k^{-1})$ and finite if c is constant. In the situation wherein (ω/k) is unbounded, the ratio $(\omega/k)/c$ is finite in the $c = O(k^{-1})$ (i.e., $\wp$ = const) limit.

Returning to the series (11), we recall that in Section C-II, we have already obtained one class of roots, which results from setting $k \cdot B_o = 0$. These roots are of the form:

$$\omega = \omega_N^{(o)} + \sigma_N k^{2N-2} + \dots, \tag{12}$$

where $\omega^{(o)}$ is identified through $a(\omega_N^{(o)}) = iN$ (cf. Eq. C-II:1'). These roots are such that there is a doublet about every ω_o^N, so that this class represents a four fold infinity of roots.*

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* Cf. footnote p. 50.

IV-a. Magneto-Hydro-Dynamics: MHD.

We will now consider in more detail the roots that are included in the finite wave speed class, i.e.: $\omega^{(o)} = 0$. Referring to Appendix B, we recall that the technique of solution envisions an expansion in powers of k of the Master determinant $|A_{ij}|$ and the system vector X_i.

The expansion of the vector X_i explicitly involves

$$
(13) \qquad
\begin{aligned}
U &= U^{(o)} + \tilde{k}U^{(1)} + \tilde{k}^2U^{(2)} + \dots \\
J &= J^{(o)} + \dots \\
B &= B^{(o)} + \dots \\
E &= E^{(o)} + \dots \\
\rho &= \rho^{(o)} + \dots \\
Q &= Q^{(o)} + \dots \\
T &= T^{(o)} + \dots \\
\hat{T} &= \hat{T}^{(o)} + \dots ,
\end{aligned}
$$

and we recall that,

$$(13.1) \qquad \omega = \tilde{k}\omega^{(1)} + \tilde{k}^2\omega^{(2)} + \dots .$$

(Note that except for the T terms we have transformed from the indivudual flow variables to the "total fluid" variables.) We are now interested in obtaining the lowest order non-trivial form of the system of equations (1-10). To these ends we perform the expansion indicated in Appendix B. Starting with Eq. (7), we have

$$(7.1) \qquad ik \times [B^{(o)} + \tilde{k}B^{(1)} + \ldots] = \mu_o(J^{(o)} + \tilde{k}J^{(1)} + \ldots) - \frac{i\omega\tilde{k}}{c^2}(E^{(o)} + \tilde{k}E^{(1)} + \ldots) \ .$$

Equating powers of k yields

$$(14) \qquad \begin{aligned} J^{(o)} &= 0 \\ ik \times B^{(o)} &= \mu_o \tilde{k} J^{(1)} - \frac{i\omega\tilde{k}}{c^2} E^{(o)} \ . \end{aligned}$$

(We have written ω for $\omega^{(1)}$.)

In similar manner we obtain the lowest order terms of the remaining relevant Maxwell's equations. They appear as

$$(15) \qquad ik \cdot B^{(o)} = 0,$$

$$(16) \qquad k \times E^{(o)} = \omega\tilde{k}B^{(o)} \ .$$

Addition and subtraction of equations (1) and (2), respectively, yields

$$(17) \qquad Q^{(o)} = 0$$

$$(18) \qquad -\tilde{k}\omega\rho^{(o)} + \rho_o k \cdot U^{(o)} = 0.$$

The addition of the momentum equations give

$$(19) \qquad -i\omega\tilde{k} \quad {}_oU^{(o)} + ik\cdot P^{(o)} = kJ^{(1)} \times B_o \ ,$$

and the subtraction gives

$$(20) \qquad E^{(o)} + U^{(o)} \times B_o = 0.$$

Addition of equations (3) and (4) yields

$$(21) \qquad \frac{P^{(o)}}{P_o} = \frac{5}{3} \frac{N^{(o)}}{N_o} \quad ,$$

or equivalently,

$$(22) \qquad P^{(o)} = \frac{10}{3} \frac{\kappa T_o}{M} \rho^{(o)} \ .$$

To obtain the latter form* we have employed the kinetic definition $P_o = N_o \kappa T_o$ (cf. Eq. B-I: 7) and $Q^{(o)} = 0$ (cf. Eq. 17).

Subtraction of the energy equations indicates that to terms of order k, ΔT is zero, i.e., $\Delta T = k^2 \Delta T^{(2)} (1 + \ldots)$. The related frequency is a single mode and is a function of second order variables, i.e., $(\Delta T^{(2)}, \Delta n^{(1)})$.

The related lowest order terms of the above equations now appear as (calling $\tilde{k}J^{(1)}$, J; $E^{(o)}$, E; etc.),

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* Also note: $2\rho = MN + (\hat{m}-m)(Q/e)$.

$$\dot{\rho} + \rho_0 \nabla\cdot U = 0, \qquad \rho_0 \dot{U} + a_0^2 \nabla \rho = \frac{1}{\mu_0}\left(\nabla \times B + \frac{\dot{U}\times B_0}{c^2}\right)\times B_0 \tag{23}$$

$$\dot{B} = \nabla \times (U \times B_0), \qquad \nabla\cdot B = 0 .$$

We have eliminated E through $E + U \times B_0 = 0$, and J through $\nabla\times B = \mu_0 J + \dot{E}/c^2$, and $a_0^2 = \frac{10}{3}\kappa T_0/M$.*

The related dispersion relation appears as,**

$$[A'^2\cos^2\theta - v^2][v^4-(a_0'^2+ A'^2)v^2+a_0^2A'^2(1 -\frac{v^2}{c^2})\cos^2\theta] = 0. \tag{24}$$

We have written v for (ω/k), and A is the Alfven speed, $A^2 = B_0^2/\mu_0\rho_0$, and $\cos\theta = B_0\cdot k/B_0 k$. The primed speeds are such that, for instance,

$$A'^2 = \frac{c^2A^2}{c^2+ A^2} \quad . \tag{24'}$$

The above dispersion relation is referred to as MHD* (Magneto-hydrodynamics plus displacement current) in Fig. 1. The left factor includes the generalized "Alfven" waves, while the right factor includes the generalized "fast" and "slow" waves.

The pressureless MHD* dispersion relation follows from the above in the limit, $a_o^2 \rightarrow 0$. There results

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* This is the adiabatic sound speed for a two component gas with equal number densities and temperatures. With γ taken to be 5/3, the effective mass is the average: M/2. (Cf. footnote, p. 38).

** Similar relations appear in references (28, 29).

$$(25) \qquad v^2(A'^2\cos^2\theta - v^2)(v^2 - A'^2) = 0 \quad .$$

We have labeled the last factor, the "cold fast" wave.

The more familiar relation follows from Eq. (24) in the limit $c \gg [v, a_o, A]$. It appears as

$$(26) \qquad (A^2\cos^2\theta - v^2)[v^4 - (a_o^2 + A^2)v^2 + a_o^2A^2\cos^2\theta] = 0 \ .$$

In accord with the literature[1-4], this latter relation is referred to as MHD in Figure 1.

The relation (24) is seen to be 6th order in (ω/k). Together with the temperature mode discussed above, these relations indicate that seven of the fourteen roots of the system (1-10) behave as

$$(13.1) \qquad \omega = \tilde{k}\omega^{(1)} + \tilde{k}^2\omega^{(2)} + \ldots \ .$$

The behavior (i.e., explicit first order terms) of the remaining seven roots will be uncovered in the succeeding section.

IV-b. Plasma Electro-Magnetics: PEM.

We proceed now to the second class of roots* which is characterized by the expansion $\omega = \omega^{(o)} + \ldots,\ \omega^{(o)} \neq 0$. As was already mentioned, it is relevant in this limit to introduce the parameter $\wp = ck$, together with the constraint $\lim\limits_{k \to o} \wp = \text{const.}$

We consider first the eliminant of Eqs. (7) and (9) and the Q,J continuity equation to obtain (cf. Eq. C-I: 27),

$$(28) \qquad \varepsilon_o E = \frac{1}{1\omega} \frac{\omega^2 J - \wp(\wp \cdot J)}{\omega^2 - \wp^2}$$

(calling $\omega^{(o)} = \omega$).

The lowest order addition and subtraction of Eqs. (5) and (6) appear as, respectively,

$$(29) \qquad \rho_o \frac{\partial}{\partial t} U = J \times B_o \ ,$$

and

$$(30) \qquad \frac{\partial}{\partial t} J - \varepsilon_o \omega_o^2 (E + U \times B_o) + \bar{\Omega}\, J \times B = -\nu_1 J \ .$$

B is a unit vector in the B_o direction.

E and U are elimated from the (transform of the) last three equations to obtain,

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[*] Many of the relations which follow, in the absence of collisions (ν, ν_1, $\nu_2 \to 0$), reduce to forms which can be found in the literature. (Cf. Ref. 10, 20, 25).

$$(31) \qquad bJ + i\,\frac{\tilde{\Omega}^2}{\omega}\,[B\times(J\times B)] + \bar{\Omega}\,J\times B - b'\wp(\wp\cdot J) = 0$$

where

$$(32) \qquad \begin{aligned} b &= \nu_1 - i\omega + \frac{i\omega\,\omega_o^2}{\omega^2 - \wp^2} = \bar{\omega} + \frac{i\omega\,\omega_o^2}{\omega^2 - \wp^2}\,, \\ \bar{\omega} &= \nu_1 - i\omega\,, \qquad\qquad b' = \frac{i\omega_o^2}{\omega(\omega^2 - \wp^2)}\,; \end{aligned}$$

$$(32') \qquad \begin{aligned} \Omega\hat{\Omega} &= \tilde{\Omega}^2 = \frac{e^2B_o^{2*}}{\hat{m}\;m}\,, \qquad \bar{\Omega}^2 = \frac{e^2B_o^2}{(\hat{m}\;m)^2}\,(\hat{m}-m)^2 = \gamma\tilde{\Omega}^2 \\ \bar{\Omega} &= \Omega - \hat{\Omega}, \qquad\qquad \gamma = \frac{(\hat{m}-m)^2}{\hat{m}\;m} \end{aligned}$$

which also serves to define the mass ratio γ.

The remaining continuity equation for ρ,U indicates that in the series $\rho = \rho^{(o)} + k\,\rho^{(1)} + \ldots$, $\rho^{(o)} = 0$, i.e., to lowest order the mass density is not perturbed. Together with the energy equations, this information gives similarly that in the related pressure series, $P^{(o)} = 0$. The subtraction of the energy equations yields

$$(33) \qquad (i\omega - 2\nu_2)\Delta T^{(o)} = 0.$$

This equation is seen to be independent of the preceeding complement. The related mode is a pure decay of the form

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* Also called lower hybrid frequency.

$$(34) \qquad \Delta T^{(o)} = \Delta T_o^{(o)}\, e^{-2\nu_2 t} \; .$$

The subscript zero indicates an initial displacement from the equilibrium situation. In addition, reference to equation (C-III: 25) indicates that $\hat{T}^{(o)} + T^{(o)} = 0$.

Returning to the above current equation, we now consider the remaining 3×3 block of the Master determinant. To evaluate it we choose a representation such that $\rho = (\rho_x, 0, \rho_z)$; $B_o = |B_o|(0,\, 0,\, B)$; so that $\rho \cdot J = \rho_x J_x + \rho_z J_z$, The secular equation appears as

$$(35) \qquad b(a^2 + \bar{\Omega}^2) - b'[ab\rho^2 + \rho_z^2(a^2 + \bar{\Omega}^2 - ab)] = 0.$$

Two other convenient forms are

$$(35') \qquad ab(a - b'\rho^2) + b\bar{\Omega}^2 - b'\rho_z^2\left[\frac{ia\tilde{\Omega}^2}{\omega} + \bar{\Omega}^2\right] = 0 \; ,$$

and

$$(35'') \qquad (a^2 + \bar{\Omega}^2)(b - b'\rho_z^2) - abb'(\rho^2 - \rho_z^2) = 0 \; ,$$

and finally, in the form of Astrom(39) and Allis(10),

$$(35''') \qquad \tan^2\theta = \frac{(b'\rho^2 - b)(a^2 + \bar{\Omega}^2)}{b[a(a - b'\rho^2) + \bar{\Omega}^2]} \; ,$$

where θ is the angle between ρ and B_o.

The variable $\mathfrak{a}$ is given by

$$\mathfrak{a} = b + i\,\frac{\tilde{\Omega}^2}{\omega} \quad , \tag{36}$$

so that

$$\begin{aligned} \mathfrak{a} - b'\wp^2 &= \bar{\omega} + \frac{i(\tilde{\Omega}^2 + \omega_o^2)}{\omega} \\ b - b'\wp^2 &= \frac{(i\omega_o^2 + \omega\bar{\omega})}{\omega} \quad . \end{aligned} \tag{37}$$

Substituting the values of $\mathfrak{a}$, b, and b' into the above gives the explicit form

$$\begin{aligned} &[i\omega^2\omega_o^2 + (\omega^2 - \wp^2)(\omega\bar{\omega} + i\tilde{\Omega}^2)][\omega(\omega\,\bar{\omega} + i(\Omega^2 + \omega_o^2))(\bar{\omega}(\omega^2 - \wp^2) \\ &\quad + i\omega\,\omega_o^2) + \omega_o^2\wp_z^2\tilde{\Omega}^2] + \bar{\Omega}^2\omega^2(\omega^2 - \wp^2)[(\omega^2 - \wp^2)\bar{\omega}\,\omega \\ &\quad + i\omega_o^2(\omega^2 - \wp_z^2)] = 0. \end{aligned} \tag{38}$$

This dispersion relation is 10th order in ω (if $\nu_1 = 0$, then it is 5th order in ω^2). Together with the temperature mode, Eq. (33), and the seven modes uncovered in Section C-IV-a, these frequencies comprise a set of eighteen roots. Since the original relation was seen to be 14th order and all of the roots fall into either one of the two categories considered, it follows that four of them are covered twice. These four roots will be exhibited in the course of examining some fundamental limiting cases.

(1) Rare-Plasma Dynamics:

This situation is obtained by letting $\omega_o^2 = e^2 n_o / \varepsilon_o \mathcal{M}$ tend to zero, to obtain

$$(\omega^2 - c^2k^2)^2 \omega\, \bar{\omega}[(\omega\, \bar{\omega} + i\tilde{\Omega}^2)^2 + \bar{\Omega}^2\omega^2] = 0. \tag{39}$$

The first factors are the vacuum electrodynamic roots (VE in Fig. 1). The second factor is degenerate, while the third is a pure decay, $i\omega = \nu_1$. The last factor yields the four decaying Larmor modes ($\overline{\text{LR}}$ in Fig. 1):

$$\begin{aligned} i\omega_{\pm}{}^{+} &= \nu_1 + i\bar{\Omega} \pm \sqrt{(\bar{\Omega} + i\nu_1)^2 + 4\tilde{\Omega}^2} \\ i\omega_{\pm}{}^{-} &= \nu_1 - i\bar{\Omega} \pm \sqrt{(-\bar{\Omega} + i\nu_1)^2 + 4\tilde{\Omega}^2} \quad . \end{aligned} \tag{40}$$

If we impose the additional constraint, $\nu_1 = 0$, Eq. (39) goes to

$$(\omega^2 - c^2k^2)^2 \omega^2 (\omega^2 - \Omega^2)(\omega^2 - \hat{\Omega}^2) = 0 \tag{41}$$

where we have transformed back to $\Omega, \hat{\Omega}$ from $\tilde{\Omega}, \bar{\Omega}$, through

$$\begin{aligned} \bar{\Omega} &= \Omega - \hat{\Omega} \\ \tilde{\Omega}^2 &= \Omega\, \hat{\Omega} \end{aligned} \tag{41'}$$

so that

$$(41'') \qquad \bar{\Omega}^2\omega^2 - (\tilde{\Omega}^2 - \omega^2)^2 = (\Omega^2 - \omega^2)(\omega^2 - \hat{\Omega}^2).$$

(2) Isotropic Plasma Dynamics.

The limit we now consider is that of a vanishing magnetic field: $\tilde{\Omega}, \bar{\Omega} \rightarrow 0$, wherein we expect to find the classical interaction of the plasma with the VE mode, i.e., $-\omega^2 + c^2k^2 + \omega_o^2 = 0$. In this limit, equation (38) reduces to

$$(42) \qquad \omega^2[i\omega\,\omega_o^2 + \bar{\omega}(\omega^2 - c^2k^2)]^2(\omega\,\bar{\omega} + i\omega_o^2) = 0.$$

The middle factor yields the dispersion relation for the phase velocity $v = \omega/k$ (MI, in Fig. 1. See below, Section 4-b, for further discussion),

$$(43) \qquad v^2 = \frac{c^2}{1 - \frac{\omega_o^2}{\omega(\omega - i\nu_1)}} \quad .$$

It may also be written as a dissipative perturbation on the more familiar coupling mode, viz.,

$$(44) \qquad \omega^2 = \omega_o^2 + c^2k^2 + i\nu_1\left[\frac{\omega^2 - c^2k^2}{\omega}\right] .$$

The explicit form of the roots of the related cubic are unrevealing, however, it is significant that one of them becomes degenerate in the limit of no dissipation. This remaining root behaves as:

$$(44.1) \qquad i\omega \sim \nu_1 c^2k^2/(\omega_o^2 + c^2k^2) \quad ,$$

for ν_1 small. More precisely, since the original factor is squared, two non-zero roots degenerate to $\omega^2 = 0$.

The last factor is a decaying plasma oscillation with the corresponding roots,

$$(45) \qquad 2\omega = -i\nu_1 \pm 2\omega_o \sqrt{1 - (\nu_1/2\omega_o)^2} \quad ,$$

so that the purely oscillatory part ω_o is decreased by the factor $\sqrt{1 - (\nu/2\omega_o)^2}$.

If the constraint $\nu_1 = 0$ is imposed, the relation (42) reduces to

$$(42') \qquad \omega^4(\omega_o^2 + c^2k^2 - \omega^2)^2(\omega_o^2 - \omega^2) = 0 \ .$$

The last factor represents pure plasma oscillation, while the second represents the interaction spoken of above.

It is interesting to follow these modes as we pass from one limit to the next. For instance, comparison of Eqs. (41) and (42') indicates that the two $\omega^2 = 0$ roots of Eq. (41) have gone to $\omega_o^2 = \omega^2$, while the magnetic modes became the degenerate $\omega^4 = 0$ modes of equation (45'). Lastly the pure VE mode $(\omega^2 - k^2c^2)^2 = 0$ became coupled with the plasma in the above limit and went to $(\omega_o^2 + k^2c^2 - \omega^2)^2 = 0$.

(3) Pressureless Plasma Dynamics.

We now proceed to examine the limiting situation $\omega_o^2 \rightarrow \infty$. The motivation of this investigation arises from our starting equations (29) and (30). Perhaps the most drastic difference between the equations of MHD (Eq. 23) and these of PEM lies in the Ohm's law equations. While in the former case (viz., $E + U \times B_o = 0$), it serves to relate E to U immediately, in the latter case it is seen to be a full equation of motion for the current. Nevertheless, comparison of these two equations reveals that they coalesce in the limit $\omega_o^2 \rightarrow \infty$. The momentum equation of this PEM limit, Eq. (29), is equivalent to the momentum equation of MHD, Eq. (19) with the pressure term removed. It follows that in the considered limit $\omega_o^2 \rightarrow \infty$, the above relation (38) should reduce to the form obtained from the MHD relation (25) in the limit $a_o^2 = 0$, viz.,

$$(25) \qquad v^2(v^2 - A'^2\cos^2\theta)(v^2 - A'^2) = 0 \; .$$

More precisely, we note that the Alfven speed A^2 remains in the formulation in the limit $\omega_o^2 \rightarrow \infty$. It will be recalled that these parameters are such that $\omega_o^2 \sim n_o$ and $A^2 \sim B_o^2/n_o$

so that in order to maintain a finite A^2 as ω_o^2 becomes large, the impressed magnetic field must also become large. The limit we consider is $\omega_o^2 \rightarrow \infty$ along the path A^2 = constant. This constraint is manifested in the constancy of the parameter, $\beta^2 = A^2/c^2 = \tilde{\Omega}/\omega_o^2$, which is now introduced. To uncover the nature of the roots about large plasma frequency, we rewrite equation (38) as,*

$$
\begin{aligned}
(46) \qquad & [\omega^2(1+\beta^2) - \beta^2\wp^2][\wp_z^2\beta^2 - \omega^2(1+\beta^2)] \\
& - \varepsilon\omega^2\Big\{(\omega^2-\wp^2)[\wp_z^2\beta^2 - \omega^2(1+\beta^2)] - [\omega^2(2+\beta^2) \\
& - \wp^2(1+\beta)][\omega^2(1+\beta^2) - \beta^2\wp^2] - \gamma\beta^2(\omega^2-\wp^2)(\omega^2-\wp_z^2)\Big\} \\
& - \varepsilon^2\omega^4\Big\{(\omega^2-\wp^2)[\omega^2(1+\beta^2) - \beta^2\wp^2] + (\omega^2-\wp^2)[\omega^2(2+\beta^2) \\
& - \wp^2(1+\beta^2)] + \gamma\beta^2(\omega^2-\wp^2)^2\Big\} + \varepsilon^3\omega^6(\omega^2-\wp^2)^2 = 0 ,
\end{aligned}
$$

where we have written ε for ω_o^{-2}, and have recalled

$$(47) \qquad \tilde{\Omega}^2 = \frac{\omega_o^2 A^2}{c^2} \; .$$

Since the related polynomial is a function of ω^2, we will seek solutions for $x = \omega^2$, of the form

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* Note that the dissipation ν_1 is set equal to zero; however, its inclusion would not effect the lowest order results herein presented.

(48) $$x = x_o + \varepsilon x_1 + \ldots\ .$$

The remaining solutions are of the form

(49) $$y = y_o\varepsilon^{-1} + y_1 + \ldots\ .$$

If these series are substituted into equation ( ), the series in x yields first

(50) $$x_o = \frac{\beta^2 \wp^2}{1+\beta^2}\ ,$$

which corresponds to the "cold" fast wave (c^2 finite);

(51) $$v^2 = \frac{A^2c^2}{c^2+A^2} = A'^2\ ,$$

where we have again written v for ω/k.

The second x_o root is

(52) $$x_o = \frac{\beta^2 \wp_z^2}{1+\beta^2}\ ,$$

or equivalently,

(53) $$v^2 = \frac{A^2c^2}{c^2+A^2}\cos^2\theta = A'^2\cos^2\theta\ .$$

Since these latter four roots (51) and (53) are identical to those extracted from the relation (24), it follows that they are the aforementioned roots which are covered twice.

(continue on p. 100a)

The series (49) yields, to lowest order, (writing y for y_o)

$$(54) \qquad y^3 - y^2(3 + \beta^2(2 + \gamma)) + y((3 + \beta^2)(1 + \beta^2) + \gamma\beta^2) - (1 + \beta^2)^2 = 0.$$

Observing that the sum of the coefficients vanishes, it follows that the related three roots are the plasma frequency $y_1 = 1$, together with

$$(55) \qquad 2y_{2,3} = 2\beta_1^2 + \gamma\beta^2 \pm \beta\sqrt{4\beta_1^2\gamma + \gamma^2\beta^2} \quad ; \quad \beta_1^2 = 1 + \beta^2 \; .$$

In terms of the relevant physical parameters, these roots appear as

$$(56) \qquad 2\omega_\pm^2 = \Omega'^2 + 2\omega_o^2 \pm \bar{\Omega}\sqrt{\Omega^{*2} + 4\omega_o^2} \quad ,$$

where

$$(56.1) \qquad \Omega'^2 = \frac{e^2B_o^2}{(\hat{m}m)^2/(\hat{m}^2 + m^2)}$$

and

$$(56.2) \qquad \Omega^{*2} = \frac{e^2B_o^2}{[\hat{m}m/M]^2} \quad .$$

These two roots (Eq. 56) are abbreviated PMO' in Figure 1. In the limit $B_o \rightarrow 0$, these roots collapse to

$$(56.3) \qquad \omega_+^2 = \omega_o^2 \; ; \qquad \omega_-^2 = 0 \; .$$

(continue on p. 101)

If we impose the constraint $\Omega \gg \omega_o$ (we have already set $\omega_o \gg$ all other frequencies), then Eq. (56) reduces to

$$(56.4) \qquad \omega_+^2 = \hat{\Omega}^2 \quad , \qquad \omega_-^2 = \Omega^2 \ ,$$

the pure Larmor modes.

To recapitulate, in the limit $\omega_o^2 \longrightarrow \infty$, A^2 = constant, the secular equation (38) tends to

$$(57) \qquad (\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)(\omega^2 - \omega_3^2)(\omega^2 - \omega_4^2)(\omega^2 - \omega_5^2) = 0 \quad ,$$

where

$$(58) \qquad \begin{gathered} \omega_1^2 = \frac{\beta^2 \mathcal{P}^2}{1 + \beta^2} \quad , \qquad \omega_2^2 = \frac{\beta^2 \mathcal{P}_z^2}{1 + \beta^2} \ , \\ \omega_3^2 = \omega_o^2 \ , \qquad \omega_4^2 = y_2 \omega_o^2 \ , \qquad \omega_5^2 = y_3 \omega_o^2 \ . \end{gathered}$$

Having determined the x_o coefficient in the series (48), the next order term x_1 is evaluated by substituting one of the x_o values, say first the Alfven mode (52) into equation (47) and again setting $\varepsilon = 0$. The lowest order terms now yield

$$(59) \qquad \begin{aligned} x_1 = \frac{x_o \mathcal{P}^2}{\beta_1^2} \Bigg\{ &- \beta_1^2 + \frac{\beta^2}{\beta_1^2} (1 + \beta_1^2) \cos^2\theta \\ &+ \frac{\gamma}{\beta_1^4} \operatorname{ctn}^2\theta (\beta^2 \cos^2\theta - \beta_1^2) \Bigg\} , \end{aligned}$$

which corresponds to the dispersion relation,

$$(59') \qquad v^2 = \frac{A^2c^2}{A^2+c^2}\cos^2\theta[1 - \frac{k^2c^2}{\omega_o^2\beta_1^2}\left\{\beta_1^2 - \frac{\beta^2}{\beta_1^2}(1+\beta_1^2)\cos^2\theta - \frac{\gamma}{\beta_1^4}\,\mathrm{ctn}^2\theta(\beta^2\cos^2\beta - \beta_1^2)\right\}] \, .$$

For $c \gg A$, this reduces to

$$(60) \qquad v^2 \sim A^2\cos^2\theta\left\{1 - \frac{k^2c^2}{\omega_o^2}(1 + \gamma\,\mathrm{ctn}^2\theta)\right\} \, .$$

Secondly, we consider the extension of the fast wave roots (50). Again, if the series (48) is substituted into equation (47) with $x_o = \beta^2\rho^2/1 + \beta^2$, the lowest order term now yields

$$(61) \qquad x_1 = \frac{x_o}{1+\beta^2}(x_o - \rho^2)[1 + \frac{\gamma}{1+\beta^2}(\frac{\beta^2\sin^2\theta - \cos^2\theta}{\sin^2\theta})]$$

which corresponds to the dispersion relation

$$(62) \qquad v^2 = \frac{A^2c^2}{A^2+c^2}\left\{1 - \frac{k^2c^2}{\beta_1^4\omega_o^2}(1 + \frac{\gamma}{\beta_1^2}(\frac{\beta^2\sin^2\theta - \cos^2\theta}{\sin^2\theta}))\right\} ,$$

which, in the limit $\beta^2 \ll 1$, reduces to

$$(63) \qquad v^2 \sim A^2(1 - \frac{k^2c^2}{\omega_o^2}(1 - \gamma\,\mathrm{ctn}^2\theta)) \, .$$

The singularities about $\theta = 0$ in equations (59-63) are ficticious and are due to the fact that when $\varepsilon = 0$, $x = k^2A^2$ is a double root of equation (47) at $\theta = 0$. It follows that the above forms are not appropriate about these values. An expression (for $\beta \ll 1$), good to $O(\varepsilon^{1/2})$, which is valid in the whole domain is obtained by completing the square of the lowest order term in (46) and then changing variables from w to $y = w - k^2A^2(\mu + 1)/2$, where $\mu = \cos^2\theta$, and $w = \omega^2/k^2A^2$. Retaining terms of order (y) and solving for w gives

$$(64) \qquad w_{\pm} = \frac{1+\mu}{2} \pm \sqrt{\left(\frac{\mu-1}{2}\right)^2 + \frac{\varepsilon'}{2}(\mu+1)(\mu\tilde{\gamma}-1)} + \frac{\varepsilon'}{2}(\mu\tilde{\gamma}-1)$$

where

$$\tilde{\gamma} = \gamma + 1$$

$$\varepsilon' = k^2c^2/\omega_o^2 \quad .$$

The (+) root is the extension of the Alfven waves, while the (-) root is the extension of the cold fast waves. In the direction $\theta = 0$, Eq. (64) indicates that the two speeds are displaced by

$$(64.1) \qquad \left.\frac{v_{fast} - v_{alf}}{A}\right|_{\theta=0} = 2\sqrt{\varepsilon'\gamma} = 2\,\frac{kc}{\omega_o}\,\frac{(\hat{m}-m)}{\sqrt{\hat{m}\,m}}$$

If $\hat{m} = m$, then the two waves again coalesce at $\theta = 0$.

The forms (60) and (63) are a more accurate description of waves in the direction $\theta \approx \pi/2$. In this direction the cold fast wave is decreased by the factor $(1 - k^2c^2/2\omega_o^2)$.

(4) Directional Waves.

(4-a): $k^2 = k_z^2 \,, \quad (k \times B_o = 0)$.

Substituting these values into the form (35) yields

$$F_+F_-(i\omega_o^2 + \omega\,\bar{\omega}) = 0, \tag{65}$$

where

$$F_\pm = (k^2c^2 - \omega^2)[\omega \mp \hat{\Omega})(\omega \pm \Omega) + i\nu_1\omega] + \omega^2\omega_o^2 \,, \tag{66}$$

and we have recalled that

$$\omega^2 \pm \bar{\Omega}\omega - \tilde{\Omega}^2 = (\omega \mp \hat{\Omega})(\omega \pm \Omega) \,. \tag{66.1}$$

(Note that these Ω's are all positive.)

The root $F_\pm = 0$, yields the dispersion relation

$$v^2 = \frac{c^2}{1 - \dfrac{\omega_o^2}{(\omega \mp \hat{\Omega})(\omega \pm \Omega) - i\nu_1\omega}} \,. \tag{67}$$

In the limit of $\hat{m} >> m$ and $\nu_1 = 0$, this reduces to the more familiar relation,(16)

$$v^2 = \frac{c^2}{1 - \dfrac{\omega_o^2/\omega}{(\omega \mp \Omega)}} \tag{68}$$

The remaining root $\omega \bar{\omega} + i\omega_o^2 = 0$, is the same root as was found in the vanishing magnetic field limit (case (2): Pure Plasma Dynamics), and is the last factor of equation (42).

(4-b): $\underline{k_z^2 = 0 \ , \quad (k \cdot B_o = 0)}$.

In this limit the form (35) reduces to

$$(69) \qquad \omega[\bar{\omega}(\omega^2 - k^2c^2) + i\omega\, \omega_o^2]\Big\{[(\omega^2 - k^2c^2)(\omega\, \bar{\omega} + i\Omega^2) + i\omega^2\omega_o^2][\omega\, \bar{\omega} + i\Omega^2 + i\omega_o^2] + \omega^2\bar{\Omega}^2(\omega^2 - k^2c^2)\Big\} = 0 \ .$$

The first two factors are independent of the magnetic field, and are associated with currents flowing parallel to the magnetic field, (indeed, they have already been encountered in the $B_o = 0$ limit, see above Eq. 42). Specifically, the non-degenerate second factor is a generalization of the so called magneto-ionic root of the ionosphere theory.^{25}

The remaining sixth order relation represents the coupling of the plasma to the magnetic field, and yields the following dispersion relation for the phase velocity (ω/k):

$$(70) \qquad v^2 = \frac{c^2}{1 + \dfrac{i\omega_o^2\omega\, \bar{\omega}}{W}}$$

where

$$(71) \qquad W = (\omega\, \bar{\omega} + i\tilde{\Omega}^2)(\omega\, \bar{\omega} + i\tilde{\Omega}^2 + i\omega_o^2) + \omega^2\, \bar{\Omega}^2 \quad .$$

In the limit of $\nu_1 = 0$,

$$(72) \qquad w = (\omega^2 - \omega_+^2)(\omega^2 - \omega_-^2) \ ,$$

$$(73) \qquad 2\omega_\pm^2 = 2\tilde{\Omega}^2 + \bar{\Omega}^2 + \omega_o^2 \pm \sqrt{\bar{\Omega}^2(\bar{\Omega}^2 + 4\tilde{\Omega}^2) + \omega_o^2(2\bar{\Omega}^2 + \omega_o^2)} \ .$$

The presence of the plasma frequency ω_o in these roots indicates a more introcate coupling between plasma and magnetic field than we observed in the $k^2 = k_z^2$ case, which is as expected.

In the limit of $\omega_o \rightarrow 0$, the above four roots go into the pure Larmor modes.

(5) Non-Propagating Waves.

In this final limit we consider $\rho \rightarrow 0$ along with k, so that the resultant phase speeds far exceed c. The determinant (35) reduces to

$$(74) \qquad \omega^4(\omega\,\bar{\omega} + i\omega_o^2)[(\omega\,\bar{\omega} + i\tilde{\Omega}^2 + i\omega_o^2)^2 + \bar{\Omega}^2\omega^2] = 0$$

The first two factors were already encountered in the first class of vacuum electrodynamics. The roots of the remaining quartic appear as,(23)

$$(75) \qquad 2\omega_\pm{}^+ = -i\nu_1 - \bar{\Omega} \pm \sqrt{(i\nu_1 + \bar{\Omega})^2 + 4(\omega_o^2 + \tilde{\Omega}^2)}$$

$$2\omega_{\pm}^{\;-} = -i\nu_1 + \bar{\Omega} \pm \sqrt{(i\nu_1 - \bar{\Omega})^2 + 4(\omega_o^2 + \tilde{\Omega}^2)}$$

where we recall that the augmented Larmor frequencies $\bar{\Omega}$ and $\tilde{\Omega}$ are related to the pure Larmor frequencies through

(41')
$$\bar{\Omega} = \Omega - \hat{\Omega}$$
$$\Omega^2 = \Omega\,\hat{\Omega} \quad .$$

All four of these modes are decaying, with a dissipative relaxation time, which for $\nu_1^2 << \omega_o^2$, is $(\nu_1/2)^{-1}$. The oscillatory part of the frequency in this approximation is given by Re(ω),

(76)
$$\mathrm{Re}(\omega^+) = -\bar{\Omega} \pm \sqrt{\Omega^{*2} + 4\omega_o^2}$$
$$\mathrm{Re}(\omega^-) = +\,\bar{\Omega} \pm \sqrt{\Omega^{*2} + 4\omega_o^2}$$

where we recall that Ω^* includes the reduced mass $\mathcal{M}$, i.e.,

(56.2) $\quad \Omega^* = eB_o/\mathcal{M}$.

and similarly for the plasma frequency ω_o,

(78) $\quad \omega_o^2 = e^2 n_o/\varepsilon_o \mathcal{M}$.

D. Conclusions.

I. General Statements.

We have seen the dissipative plasma oscillatory phenomena emerge from a self-contained microscopic analysis. This analysis indicated that there are two fundamental regions of excitation. These regions are governed by two distinct types of dispersion relations: $\omega = k\omega^{(1)} + k^2\omega^{(2)} + \ldots$, and $\omega = \omega^{(o)'} + k\omega^{(1)'} + k^2\omega^{(2)'} + \ldots$, $\omega^{(o)'} \neq 0$, respectively. The first region is indicative of finite wave propagation, ($\lim\limits_{k\to o} (\omega/k) = \text{constant}$), and yields Lundquist's Magneto Hydro-dynamics (MHD), wherein the plasma exhibits the fluid dynamical property of sound propagation. In the second region, $\lim\limits_{k\to o} (\omega/k)$ is unbounded. In this domain of phenomena, the plasma acts cooperatively and in an ordered manner with respect to the external B_o field, in addition to exhibiting an interaction of these oscillations with the electromagnetic fields. These Plasma Electro-Magnetic (PEM) phenomena are also characterized by fluctuations whose time variations far exceed their spatial variations. The wave number k was kept in the lowest order (viz., $k \to 0$) terms by introducing the parameter $\rho = ck$. If this parameter is kept finite while all other space derivatives are set equal to zero, then the analysis includes time variations which are large compared to only the space variations originating in the particle dynamics part of the system (i.e., Boltzmann equations). In the MHD region, space and time derivatives are seen to be of the same order.

The lowest order (in k) form of the master determinant was seen to be 14th order in ω. In the above scheme of ordering these roots, seven emerge in the MHD limit, while eleven emerged in the PEM limit. The four roots which were covered twice were discovered to be the cold fast waves and the Alfven waves.

In terms of the equations governing the two regions, in the momentum equation the distinction resides in the gradient of pressure term, which vanishes in the PEM limit but is finite in the MHD*† limit. A sufficient condition that induces such a separation is that the equilibrium temperature T_o be either zero or finite respectively. The distinction between the two current equations is that in the MHD case it is the simple vector relation $E + U \times B_o = 0$, while in the PEM region it is a full equation of motion for the current. Having observed that in the latter equation the plasma frequency multiplies the $E + U \times B_o$ term, it follows that the PEM system collapses to the pressureless MHD*‡ as $\omega_o \rightarrow \infty$ (together with the constraint $B_o \rightarrow \infty$). This was seen to be so, with the modified Alfven speed $A'^2 = c^2 A^2/(c^2 + A^2)$ appearing in place of the standard $A^2 = B_o^2/\mu_o \rho_o$. The Alfven and fast waves were continued to terms of $O(\omega_o^{-4})$, and a velocity spread in the parallel direction (i.e., to B_o) between the Alfven and fast waves was found. This increment vanishes in the limit of equal masses. In the normal direction the cold fast wave was found to suffer a loss in velocity (cf. Eq. C-IV: 63).

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† These statements are made for the related first order (in k) theory.

‡ The star superscript on the MHD abbreviation indicates that the displacement current is included.

The fact that these waves are recaptured in the limit $k = 0$, $\wp \neq 0$, is indicative of their electromagnetic origin.

In the PEM limit the energy equations* yield a decaying finite temperature difference to the equilibrium value $\Delta T = 0$, with a related decay time $(2\nu_2)^{-1}$. However, in the MHD class ΔT is zero to terms of order K. The addition of the MHD energy equations yield a sound speed $a_o^2 = \frac{10}{3} kT_o/M$.**

The PEM roots in the limit of vanishing number density went to one degenerate $\omega = 0$, a purely decaying mode $i\omega = \nu_1$; the vacuum electrodynamic roots $(\omega^2 - p^2)^2 = 0$, and four decaying Larmor roots (cf. Eq. C-IV: 39). The fact that the Larmor roots remain in the limit of vanishing number density, while the plasma oscillations collapse to zero is indicative of the fact that the latter effect is a cooperative one, while the Larmor resonances are, as already stated, an ordered response to an external influence.

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* The terminology is more convenient than accurate. A more precise name would be "non-adiabatic equations".

** Note: In limit $\hat{m} = m$, $M = 2m$ and $a_o^2 \rightarrow \frac{5}{3} kT_o/m$; cf. footnote, p. .

In the opposite extreme of vanishing B_o field the plasma acts collectively, yielding decaying plasma oscillations with a decay frequency, $\mathrm{Im}\ 2\omega = -\nu_1 \pm \mathrm{Im}\sqrt{4\omega_o^2 - \nu_1^2}$ and also exhibits an interaction of this collective behavior with the electromagnetic fields to yield six decaying roots (cf. Eq. C-IV: 42). If the dissipation is set equal to zero, then two of these roots degenerate to $\omega = 0$, while the remaining four to to the classical magneto-ionic relation $(\omega^2 - \omega_o^2 - k^2c^2)^2 = 0$. The decaying plasma oscillations become purely oscillatory.

As expected, the decaying plasma oscillations are also present in waves which are directed parallel to the magnetic field. In addition to this second degree root, the remaining eighth order root in this direction splits into two fourth order roots, one being obrained from the other through the reflexion: $+\Omega \rightarrow -\Omega$, $-\hat{\Omega} \rightarrow +\hat{\Omega}$. The form of these roots, say $F_+ = \left\{(k^2c^2 - \omega^2)[(\omega-\hat{\Omega})(\omega+\Omega)] + \omega^2\omega_o^2\right\}$, $(\nu_1 = 0)$, indicates how the Larmor resonance decouples the plasma oscillations from the vacuum roots $\omega^2 - k^2c^2$. If this root is inverted and solved for $v^2 = \omega^2/k^2$, one sees the genesis of the magnetic part of the MHD roots; for in the limit of n_o, $B_o \rightarrow \infty$ (again writing $\tilde{\Omega}^2 = \omega_o^2\beta^2$), we see that $v^2 \rightarrow A'^2$. The other F_- factor gives the other $v^2 = A'^2$, so that both the fast waves and the Alfven waves are recaptured in the parallel direction.

The decaying magneto-ionic roots of the vanishing magnetic field limit are also present in waves directed perpendicular to the magnetic field ($k \cdot B_o = 0$). The remaining sixth order root in this direction yields only the fast wave $v^2 = A'^2$ in the MHD limit. This is, of course, as expected since the Alfven waves vanish normal to the magnetic field.

The totally non-propagating limit $k = 0$, $\rho = 0$, yields purely oscillatory modes, the most interesting of which are four mixed plasma-magnetic fluctuations (cf. Eq. C-IV: 75). Besides a degenerate $\omega^4 = 0$, the only other mode contains the two decaying plasma oscillations. It should be noted that these roots are the formal lowest order roots in the expansion $\omega = \omega^{(o)} + k\omega^{(1)} + \ldots$, with all k (including those originating in Maxwell's equations) explicit. Although such roots are associated with a rigidly homogeneous medium, the non-isotropic structure of the system is present. It manifests itself in the related system eigenvector X_i (cf. Eq. C-I: 22), the pertinent components of which are $(J_{\parallel}, J_{\perp})$. It is readily established that the plasma oscillations are associated with the vector $(J_{\parallel}, 0)$ while the remaining mixed plasma-magnetic oscillations are eigenvalues with the related eigenvector $(0, J_{\perp})$.

More generally, as mentioned in the introduction, in any of the above limits wherein $B_o \neq 0$, those modes which do not contain the magnetic field are the eigenvalues of the eigenvector $(J_{\parallel}, 0)$, while the complementary non-degenerate roots are the eigenvalues of $(0, J_{\perp})$.

The non-isotropic nature of the system becomes evident at an early stage of the analysis, and is manifested in the distinctly different nature of the perpendicular and parallel components of the wave number, $k_{\perp}$ and $k_{\parallel}$, respectively. This non-isotropic quality of the problem permitted the Larmor resonance phenomena to be examined in the limit of $k_{\parallel} = 0$, while terms of $o(k_{\perp}^{2N-2})$ were retained (N = order of resonance). This class of roots is four-fold infinitely denumerable,* and is seen to involve $O(k_{\perp}^{2N-2})$ variations above the classical integral multiples of the Larmor frequencies. The first order (in k) decay time is as the inverse self-collision frequency, and is independent of the resonance order of the mode. The dispersion coefficient includes the factor τ (cf. Eq. C-II: 30). The form of this coefficient indicates that the dispersion vanishes as $B_0 \rightarrow \infty$, while if $n_0 \rightarrow \infty$ the dispersion is seen to be enhanced. This τ factor also includes the influence of the "other gas" on the dispersion of the Larmor resonances of the first gas (e.g., that of the ions on the electron resonances). In the absence of dissipation ($\nu = 0$), this factor (viz., $(a(\omega_N)-1)$ for the case of ion resonance) appears as $N + m/\hat{m}$ which does not vanish. This is indicative of the difference in charge of the two species, so that for any given N the charges are rotating in opposite senses.*

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* More precisely, including the image singularities about $a = -iN$, this class constitutes an eight-fold infinity of roots.

The connection between the MHD and PEM regions and subregions is depicted in Figure 1.

Returning to the conclusions of the general analysis, in the formulation of the transfer equations, non-local (i.e., in time) terms were obtained which collapsed to the more standard local forms after a few collision times. More directly the related coefficients are such that they become constant for values of the self-collision frequencies which far exceed the phenomena frequency ω. In addition, when the moment equations were constructed from the stated model, a technique of generating higher order variables was uncovered, according to which the J-flux tensor R_{ij} was defined (cf. Sec. B-II-c, and App. F). This latter conclusion was made evident by the lack of constraints imposed upon the mass ratio.

In this vein the parameter $\bar{\Omega}$ is significant. We recall that it is the difference of the two absolute Larmor frequencies so that if $m = \hat{m}$, $\bar{\Omega}$ vanishes. The difference between $\bar{\Omega} = 0, \neq 0$ has manifestations first in the current equation wherein we find that if $\bar{\Omega} = 0$, the Hall current term vanishes. Secondly, in the PEM dispersion relation, the factors which go over to the MHD roots are also obtained if the constraint $m = \hat{m}$ is included. Physically, the difference in mass term imposes an inertial asymmetry in the system. In the next order in $\varepsilon = \omega_o^{-2}$, wherein the $\bar{\Omega}$ term enters, the cold fast wave is seen to loose its isotropic structure (cf. Eq. C-IV: 63) and, in addition, if $\hat{m} = m$ the cold and Alfven waves again coalesce in the parallel direction.

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* If the two charges were of the same sign, then the dispersion of the $(\hat{m}/m)$th order ion resonance would vanish, due to the support of the first order electron$^{(+)}$ resonance.

In this latter part of the conclusion section perhaps it is appropriate to state some rules of thumb, so to speak, pertaining to the relevant equations and dispersion relations of plasma physics. With regard to the governing macroscopic equations we have noted that in the MHD theory, of the two equations -- the momentum and current equations -- the former is the significant equation of motion, while in the PEM domain the current equation is the relevant equation of motion. Recalling that these two equations are equally significant since they are completely equivalent to the two individual-component momentum equations (and, indeed, come about from a change of variables from individual flow variables to total vluid variables), it is not surprising that if two theories evolve from this system, and in one of them, say, the addition equation (i.e., the "moment" equation) is significant, then certainly the other theory should describe phenomena governed by the difference equation (Ohm's law).

We have also noted that the wave number enters differently into the particle dynamics part of the system (Boltzmann equations) as compared to its roll in the field dynamics part of the system (Maxwell's equations). In the latter domain it was natural and revealing to change variables from k to $\not{p} = ck$, which, in addition, insured that c was of proper order of magnitude (viz., $O(\varepsilon^{-1})$). Something may also be said of the dual location of ω. For the most part we have seen that the dissipation varameters ν and ν_1 appear in the form $\nu - i\omega$ and $\nu_1 - i\omega$. It is observed that this is so for only those ω's which originate in the Boltzmann part of the system. Factors of the form ω, $\omega^2 - \not{p}^2$, ..., have their origins in the field equations.

D-II. Regions of Validity.

The asymmetry in the perpendicular and parallel (i.e., to B_o) parts of the formulation was immediately revealed by the expansions involved in the analysis. These were seen to divide into two distinct classes.

Firstly, the expansions about $k_{\parallel}$ were present in the development of the $S^N C^M$ functions. These forms are functions of $k_{\parallel}$ through the parameter $\bar{a} = (\nu - i\omega + ix)/\Omega$, where $x = k_{\parallel}\xi_{\parallel}$. The related expansions involve terms of the form

$$\bar{a}^{-1} = \frac{\Omega}{\nu - i\omega}\left(1 - \frac{ik_{\parallel}\xi_{\parallel}}{\nu - i\omega} + \ldots\right).$$

The expansion parameter is $\frac{C}{\lambda(\nu - i\omega)}\, x = \varepsilon x$, where x is non-dimensional through the wavelength λ, and the thermal speed C, and is of $O(1)$. In the MHD limit, $\nu \gg \omega$, so that $\varepsilon \sim \lambda_c/\lambda$, where λ_c is the mean free path, $\lambda_c \sim C/\nu$. In this limit the related series in x is seen to converge rapidly if $\lambda \gg \lambda_c$.

In the PEM limit, wherein ω remains finite as $k \rightarrow 0$, it is relevant to consider $\omega \gg \nu$, in which case $\varepsilon \sim C/v$, (where $v = \lambda\omega$), so that the convergence condition becomes $v \gg C$, which in this limit is usually the case.

The second class of expansions are centered about small $k_\perp$, and appear in the development of the exponential form

$$e^{ik_\perp \xi_\perp/\Omega} = 1 + i\,\frac{k_\perp \xi_\perp}{\Omega} + \ldots .$$

The relevant expansion parameter is $k\xi/\Omega = Ly/\lambda$, where y is non-dimensionalized through the Larmor radius $L = C/\Omega$, and the thermal speed C, so that y is of $O(1)$. The condition for rapid convergence now appears as $\lambda \gg L$.

The criterion to be satisfied in order that the above sufficient conditions (viz., $\lambda \gg L, \lambda_C$) be realized, is that these lengths be small compared to the dimensions of the environment.* To these ends we will now compute the mean free path and the Larmor radius for a few relevant physical environments. All of the following are order-of-magnitude calculations. The mass involved is taken to be 10^{-24} gms., ($\sim$ proton mass).

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* We note that these are only sufficient conditions. Indeed, many of the results which were herein obtained have been observed in the laboratory where λ_C, in most instances, is gigantic compared to the apparatus lengths.

The mean free path λ_C is given, approximately, by

$$\lambda_C \approx n_o d^4 .$$

The distance d is the Debye distance

$$d^2 = \frac{\varepsilon_o \kappa T_o}{e^2 n_o} = \frac{c^2}{\omega_o^2} , \qquad d(\mathrm{cm}) \approx 7(T_o/n_o)^{1/2} ,$$

so that for λ_C we may write

$$\lambda_C \approx 3 \times 10^3 \, T_o^2/n_o \; (\mathrm{cm}).$$

The Larmor radius is given by

$$L(\mathrm{cm}) = C/\Omega \approx (3/5) T_o^{1/2}/B_o .$$

The following table has been constructed with the use of these formulas.

			The Sun (33)				
	Sherwood (16)	Interstellar Space (32)	Center	Photosphere	Chromosphere	Corona	Units
T_o	10^8	10^4-10^2	10^7	10^4	10^4-10^5	10^4-10^6	^oK
n_o	10^{15}	10^0	10^{24}	10^{17}	10^{10}	10^7	cm^{-3}
B_o	10^4	10^{-5}	10^{*}	10^{*}	10^{*}	10^{*}	gauss
λ_C	10^4	$10^{11}-10^7$	10^{-7}	10^{-6}	$10-10^3$	$10-10^8$	cm
L	10^0	10^7-10^6	10^3	10	10	$10-10^2$	cm

It follows that the related inequalities are more readily established in the astronomical regions.

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* These values become very doubtful in the inner regions.

D-III. Related Problems.

We have just noted that the included formulation was constructed about the expansion of the S^nC^m functions and the e^{iy} function. It will be noted that in the event that B_o becomes small, the latter expansion is poorly convergent.* The related formalism which includes the small B_o domain phenomena is uncovered if we recall the form of the moments I_ℓ, (cf. Eq. C-II: 1),

$$I_\ell = 2 \int \frac{\phi}{e^{4\pi\bar{a}} - 1} \oint e^{\sigma} h \, d\psi \, d^3\xi \ .$$

(The notation is self-contained.) We are now interested in the expansion of this integral about large σ. Such an expansion is generated by a parts operation of the following form:

$$\oint e^{\sigma} h \, d\psi = \oint e^{\sigma}\sigma' \frac{h}{\sigma'} \, d\psi = \oint [e^{\sigma}]'[\frac{h}{\sigma'}]d\psi = e^{\sigma} \frac{h}{\sigma'} \Big|_{0}^{2\pi} - \oint e^{\sigma}[\frac{h}{\sigma'}]' d\psi$$

In this manner we obtain the desired asymptotic series (recalling $\sigma \sim \varepsilon^{-1}$), which to third order appears as

$$\oint e^{\sigma} h \, d\psi = \left[\frac{e^{\sigma}}{\sigma'} \left\{ h - (\frac{h}{\sigma'})' + ((\frac{h}{\sigma'}) \frac{1}{\sigma'})' - \left(((\frac{h}{\sigma'}) \frac{1}{\sigma'})' \frac{1}{\sigma'}\right)' \right\} \right]_{0}^{2\pi}$$

$$+ \oint e^{\sigma} \left((((\frac{h}{\sigma'})' \frac{1}{\sigma'})' \frac{1}{\sigma'})' \frac{1}{\sigma'}\right)' \, d\psi \ .$$

---

* In the PEM investigation, this difficulty was obviated when $B_o \rightarrow 0$, since the analysis was such that λ was a higher order infinitesimal than L (i.e., it vanished first in the limiting process).

Let us consider the lowest order term,

$$\oint e^{\sigma} \mathcal{h}\, d\xi \sim \frac{\exp\{2\bar{a}\psi + 2ik_{\perp}\cdot\xi_{\perp}\sin\psi\}}{2\bar{a} + 2ik_{\perp}\cdot\xi_{\perp}\cos\psi}\, \mathcal{h}(\psi)\Big|_{0}^{2\pi} + \ldots .$$

Recalling that $\mathcal{h}$ is periodic in ψ, we obtain

$$\oint e^{\sigma}\, d\psi \sim \frac{(e^{4\pi\bar{a}} - 1)}{2\bar{a} + 2ik_{\perp}\cdot\xi_{\perp}}\, \mathcal{h}(0) + \ldots .$$

The total moment I_{ℓ} appears as

$$I_{\ell} \sim \int \frac{d^3\xi\ \not{\!\phi}\ h(\xi) H_{\ell}(\xi)}{\nu - i\omega + ik\cdot\xi} + \ldots .$$

This first term is recognized to be the ℓ^{th} moment of g with $B_0 = 0$.

Another source of remaining endeavor is found in the integrations over $\xi_{\parallel}$ (cf. p. 67). We recall that the region of investigation in the ω plane was restricted to $\mathrm{Re}(-i\omega) \neq \nu$ in order to maintain well defined integrations. It is at this point that a more intensive transform investigation would uncover the influence of the dissipative parameters and the Larmor resonances in the extended Landau problem.[8,30] It is interesting to observe that such a formulation is constructed about the parallel part of the analysis, which would imply that even these extended phenomena with an impressed magnetic field still have their origins in the electric part of the fields.

In conclusion, the standard problem remains of the extension of the related expansions about large wavelength to higher order terms, together with the complementary investigation about small wavelength, which would start along the lines of the above series in σ^{-1}.

Appendix* A: The Collision Forms.

In this appendix we will show how the postulated equations (B-II: 1,2) evolve as a first step in a certain itteration scheme solution to the full Boltzmann equations.** The procedure is such that the correction to the local Maxwellian f_o is given by

$$f^{(N)} = f_o + g_1 + g_2 + \dots + g_N \; ,$$

or equivalently,

$$f^{(N)} = f^{(N-1)} + g_N \; .$$

These correction terms are determined by the Boltzmann equations, written in the form (say, for f_+):

$$Df_+ = \int [f_+^{(N-1)'} f_{+,1}^{(N-1)'} - f_+^{(N)} f_{+,1}^{(N)}] \, d\Sigma_{++}$$

$$+ \int [f_+^{(N-1)'} f_-^{(N-1)'} - f_+^{(N)} f_-^{(N)}] \, d\Sigma_{+-} \quad .$$

The operator D represents the convective derivative,

$$D \equiv \frac{\partial}{\partial t} + \xi \cdot \nabla + (e/m)_+ \, (E + \xi \times B) \cdot \nabla_\xi$$

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* The notation in these appendices is independent of the preceeding analysis and wherever necessary is redefined.

** It has been shown that these forms also follow from an eigenfunction expansion of the linearized collision integrals (H. Grad and L. Sirovich, private communication); cf. also ref. (31), Sec. 30, p. 124, and refs. (34, 35).

and

$$d\Sigma_{ij} = d^3\xi_1^j \, |V_{ij}| d\sigma_{ij} \, .$$

V_{ij} is the relative velocity $(\xi_i - \xi_j)$. The Coulomb cross section is given by

$$V_{ij} \, d\sigma_{ij} = [e^2/4\pi\varepsilon_o \mathcal{m}]^2 (\alpha \cdot V)^{-3} d\Omega \ ,$$

where α is a unit vector that bisects V and V'. The prime and subscript 1 notation are standard (cf. references 19, 31). The element of solid angle is about the spherical angular coordinated (θ, ϕ) of the vector α, so that $(0 \leq \theta \leq \frac{\pi}{2})$.

Let us consider the first iteration, calling $f^{(1)} = f$, and dropping the (+) subscript on the functions in the self-collision terms.

$$\begin{aligned} Df_+ &= \int \left\{ f_o' \, f_{o,1}' - [f_o f_{o,1} + f_o g_1 + f_{o,1} \, g] \right\} d\Sigma_{++} \\ &\quad + \int \left\{ f_{o,+}' \, f_{o,-}' - [f_{o,+} \, f_{o,-} + f_{o,+} g_- + f_{o,-} g_+] \right\} d\Sigma_{++} \\ &= - \int \, (f_o g_1 + f_{o,1} \, g) d\Sigma_{++} + J_{+-}[f_{o,+} ; \, f_{o,-}] \\ &\quad - \int \, (f_o^+ g_- + f_{o,-} g_+) d\Sigma_{+-} \ , \end{aligned}$$

neglecting non-linear terms in g. The operator J_{+-} represents the cross collision integral.

For Maxwellian molecules we recall that $Vd\sigma$ is independent of V so that $\int g_1 d\Sigma_{++} = \int g_- d\Sigma_{+-} = 0$, as these terms include the factor $\int (f_o - f) d^3\xi = n - n = 0$, recalling that the first three moments of f_o are equal to those of f. If we continue this property to the case of Coulomb particles the above becomes

$$Df_+ = \nu^*[f_o - f] + J_{+-}[f_{o,+}; f_{o,-}] ,$$

$$\nu^* = \int f_+ d\Sigma_{++} + \int f_{o,-} \, d\Sigma_{+-} .$$

The desired form follows (i.e., with the appropriate simplifications imposed on the collision frequencies), if these collision terms are linearized about the absolute Maxwellian states $(f_+^{(o)}; f_-^{(o)})$.

Appendix B: The Formal Expansion.

The algebraic system of transform equations (C-II: 1) are written symbolically

(1) $$A_{ij} X_j = 0.$$

A is a 10 x 10 matrix (with the thirteen external parameters: $e, \varepsilon_o, \mu_o, m, \hat{m}, B_o, n_o, \kappa, T_o, \hat{\nu}, \nu, \nu_1, \nu_2$), and X is a 10 x 1 collumn vector the components of which are transform perturbations: $[n, u, T; \hat{n}, \hat{u}, \hat{T}]$.

The technique of solution is to expand these variables in a power series about $k = 0$ (where k is appropriately non-dimensionalized), so that

$$(2) \qquad \omega = \omega^{(o)} + k\omega^{(1)} + k^2\omega^{(2)} + \dots,$$

so that,

$$(3) \qquad A(k,\omega) = A(k, \omega^{(o)} + k\omega^{(1)} + \dots)$$

$$= A(0, \omega^{(o)}) + \left(A_k + A_\omega \omega^{(1)}\right)_{k=0} k + \dots$$

Similarly, for the eigenvector X we write

$$(4) \qquad X = X^{(o)} + kX^{(1)} + k^2X^{(2)} + \dots .$$

If these latter expansions are substituted into $AX = 0$, and coefficients of equal powers in k are equated, there results

$$A^{(o)}(0, \omega^{(o)})X^{(o)} = 0$$

$$A^{(o)}(0, \omega^{(o)})X^{(1)} + [A_k(0, \omega^{(o)}) + \omega^{(1)}A_\omega(0,\omega^{(o)})]X^{(o)} = 0$$

$$\vdots$$

This system may be rewritted in the form

(6) $A^{(o)} X^{(o)} = 0$

(6.1) $A^{(o)} X^{(1)} = y_1(\omega^{(1)})$; $|A^{(o)}| = 0$

(6.2) $A^{(o)} X^{(2)} = y_2(\omega^{(2)})$; $|A^{(o)}| = 0$

$$\vdots$$

The m^{th} equation appears as

(6.m) $A^{(o)} X^{(m)} = y_m(\omega^{(m)})$; $|A^{(o)}| = 0$.

The vanishing of the determinant $|A^{(o)}|$ insures the existence of non-trivial solutions to equation (6), and serves to determine $\omega^{(o)}$, whence $X^{(o)}$. This condition, $|A^{(o)}| = 0$, implies that the linear equations comprising (6) are not independent or, equivalently, that the column vectors comprising $A^{(o)}$ are linearly dependent. It follows that solutions to (6.m) will exist only when the components of the vector y_m are linearly dependent in the identical manner as are the column vectors of $A^{(o)}$. Formally, for matrix equations such as (6.m) this necessary and sufficient condition for the existence of solutions is that all the left null vectors of $A^{(o)}$ be orthogonal to y_m. (The more general theorem covering the case when $A^{(o)}$ is an arbitrary linear operator is that the right null vectors of the adjoint of $A^{(o)}$ be orthogonal to y_m.)[36]

Geometrically, if $A^{(o)}$ is of rank r and dimension n, it maps all vectors in R_n onto the hyperspace R_r, so that for $A^{(o)}X = y$ to be meaningful, y must lie in R_r. The normal vectors to this hyperspace are the left null vectors of $A^{(o)}$ (viz., if z is such that $\langle zA^{(o)}| = 0$, then $\langle z|A^{(o)}x\rangle = \langle zA^{(o)}|x\rangle = 0$).

Returning to equation (6.m), suppose γ is a left null vector of $A^{(o)}$, then solutions will exist only if

$$(6.m') \quad \langle \gamma | y_m(\omega^{(m)}) \rangle = 0 \ ,$$

which, in turn, serves to determine $\omega^{(m)}$. Consider that $A^{(o)}$ is of rank r, dimension n ($n = 10$ in the original system). There will be $(n-r)$ redundant equations in the set (6.m) when ω_m satisfying (6.m') is resubstituted. Discarding these $(n-r)$ equations, the remaining system of r equations in n unknowns yields a $(n-r)$ parameter of solutions $X^{(m)}$.

Specifically, equation (6.1') appears as,

$$-\omega^{(1)} = \frac{\langle \gamma | A_k \ X^{(o)} \rangle}{\langle \gamma | A_\omega \ X^{(o)} \rangle} \quad .$$

Resubstitution of this value into equation (6.1) yields $X^{(1)}$, so that the problem is solved to $O(k)$ terms, and appears as

$$\omega = \omega^{(o)} + k\omega^{(1)}$$

$$X = X^{(o)} + kX^{(1)} \quad .$$

The higher order corrections are similarly developed.

Appendix C: Other Methods of Integration.

In Section C-III (p.67) it was noted that the $d\xi_{\parallel}$ integrations remain bounded if $\mathrm{Re}(-i\omega) > -\nu$. An alternate procedure may be introduced in the early stages of the analysis to cover the complement of roots:* $\mathrm{Re}(i\omega) > \nu$. To these ends we recall equation (C-II: 1) for the ℓ^{th} moment of g and consider specifically the magnetic weighting factor $(e^{4\pi\bar{a}} - 1)^{-1}$. In the event that $\mathrm{Re}(\bar{a}) < 0$ (which is satisfied if $\mathrm{Re}(i\omega) > \nu$), the above factor may be expanded

$$[1 - \exp(4\pi\bar{a})]^{-1} = \sum_{n=0}^{\infty} e^{n4\pi\bar{a}} \quad ,$$

so that I_{ℓ} becomes

$$I_{\ell} = -2 \int d^3\xi\phi \int_0^{\infty} d\psi \exp\left\{2\bar{a}\psi + 2iy \sin \psi\right\} \hbar H_{\ell} \ .$$

In this scheme of integration, three of the remaining four integrations are readily performed, while the remaining integral is not expressible in terms of well known functions (except for certain specified directions of propagation).[6]

---

[*] The included analysis, of course, also covers this complementary region; cf. footnote p. 67.

Another method of obtaining a more inclusive set of formulas is obtained if we rewrite the factor $e^{2\bar{a}\psi}/(e^{4\pi\bar{a}}-1)$ in terms of its Fourier representation:

$$e^{2\bar{a}\psi}/(e^{4\pi\bar{a}}-1) = \sum_{n=-\infty}^{+\infty} \frac{e^{in\psi}}{2\pi(2\bar{a}-in)} \qquad .$$

The integral, C-II: 1, now appears as

$$I_\ell = \sum_{-\infty}^{\infty} \int d^3\xi \oint d\psi \, \frac{\emptyset \exp\{in\psi + 2in \sin\psi\}}{\pi(2\bar{a}-in)} \hbar(R_\psi \xi) H_\ell(R_{-\psi}\xi) .$$

The integrations over ψ yield linear combinations of Bessel functions (cf. p. 53).

Appendix D: Tabulation of Integrals.

In the following tabulation, we recall the definition

$$\left\langle F \right\rangle \equiv \int \emptyset F d^3\xi = \frac{1}{(2\pi)^{3/2}} \int_0^\infty \int_{-1}^{1} \int_0^{2\pi} e^{-\xi^2/2} \, F \, \xi^2 d\xi \, d\cos\theta \, d\emptyset$$

and $H_2 = (\xi^2 - 3)/2$.

$$\left\langle \xi_i \xi_j \right\rangle = \delta_{ij}$$

$$\left\langle \xi^{2n} \right\rangle = 1.3 \ldots (2n+1); \qquad \left\langle \xi_i^{2n} \right\rangle = 1.3 \ldots (2n-1)$$

$$\left\langle \xi_i^{2n} \, \xi_j^{2m} \right\rangle = \frac{1}{4} \; \xi^{2(n+m)} \quad T_{nm} \; , \quad \text{where}$$

$$T_{nm} = \int_{-1}^{1} x^{2n}(1-x^2)^m dx \; .$$

$$\left\langle \xi_i^{2m} \, \xi_j^{2n} \right\rangle = \frac{1.3 \; \ldots \; (2(m+n) + 1)}{2(2(m+n) + 1)}$$

$$\left\langle \xi^{2m} \, \xi_j^{2n} \right\rangle = \frac{1.3 \; \ldots \; (2(m+n) + 1)}{2n+1}$$

$$\left\langle H_2 \right\rangle = 0 \; ; \quad \left\langle H_2^2 \right\rangle = \frac{3}{2} \; ; \quad \left\langle \xi_i^2 \right\rangle = \left\langle H_2 \xi_i^2 \right\rangle = 1 \; ,$$

$$\left\langle H_2 \xi^2 \right\rangle = 3 \; ,$$

$$\left\langle (\xi \cdot A)\xi \right\rangle = A_\mu \left\langle \xi_\mu \xi_\tau \right\rangle = A_\mu \delta_{\mu\tau} = A$$

$$\left\langle (\xi \cdot A)(\xi \cdot C) \right\rangle = \left\langle (\xi \cdot A)(\xi \cdot C) H_2 \right\rangle = A \cdot C$$

$$\left\langle \xi_i \xi_j \xi_k \xi_\ell \right\rangle = \delta_{ij} \, \delta_{k\ell} + \delta_{ik} \, \delta_{j\ell} + \delta_{jk} \, \delta_{i\ell}$$

$$\left\langle (\xi \cdot k_\perp)^2 \xi \times B \;\, \xi \cdot B \times h_1 \right\rangle = (B \times h_1)_\nu \; k_\perp^\lambda \; k_\perp^\mu \; B_\tau \varepsilon_{\emptyset\tau\alpha} \left\langle \xi_\lambda \xi_\mu \xi_\emptyset \xi_\nu \right\rangle$$

$$= k_\perp^2 (B \times h) \times B + 2(k_\perp \times B) k_\perp \cdot (B \times h_1) \; .$$

This latter integral appears in the coefficients that multiply the $[S^4]^0$, $[S^3C]^{(0)}$, and $[C^2S^2]^{(0)}$ terms in the velocity moment equations (cf. Sec. C-III-b).

The following integration appears in Section C-II:

$$L = \left\langle (\xi \cdot V)\xi(k_\perp \cdot \xi_\perp)^{2n-2} \right\rangle = \frac{1}{(2\pi)^{3/2}} \int e^{-\xi^2/2}\ \xi \cdot V\ \xi(k_\perp \cdot \xi_\perp)^{2n-2}\ d^3\xi$$

where V is such that $B \cdot V = 0$.

Let us choose a representation wherein $k_\perp = (k_\perp, 0, 0)$ and the cylindrical coordinate of the vector ξ is measured from $k_\perp$ in the plane normal to B. It follows that

$$\xi = [\xi_\perp(\cos\alpha, \sin\alpha);\ \xi_\parallel]$$

$$\xi^2 = \xi_\perp^2 + \xi_\parallel^2$$

$$(k_\perp \cdot \xi_\perp)^{2n-2} = k_\perp^{2n-2}\ \xi_\perp^{2n-2}\ \cos^{2n-2}\alpha\ ,$$

and

$$\begin{pmatrix} L_x \\ L_y \end{pmatrix} =$$

$$\frac{k_\perp^{2n-2}}{(2\pi)^{3/2}} \int_0^\infty d\xi_\perp \int_{-\infty}^{\infty} d\xi_\parallel \int_0^{2\pi} d\alpha e^{-\xi^2/2}(V_x\cos\alpha + V_y\sin\alpha)\ \xi_\perp^{2n+1}\cos^{2n-2}\alpha \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

The three standard integrations that are included are: $\oint \cos^{2n}\alpha d\alpha = 2\pi\ b_{2n}$, $\int_0^\infty y^{2n+1}\exp(-y^2/2)dy = 2^n n!$, and $\int_{-\infty}^{+\infty}\exp(-z^2/2)dz = (2\pi)^{1/2}$. If k is a unit vector in the $k_\perp$ direction, then,

$$L(V) = 2^n n!\ k_\perp^{2n-2}[b_{2n} k\ k\cdot V + b'_{2n} k \times B(k \times B\cdot V)]$$

$$b_{2n} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdots(2n)}\ , \qquad b'_{2n} = b_{2n}(2n-1)^{-1}\ .$$

Appendix E: A Note on Determinants.

In the early stages of this investigation, it was convenient, when working with the two dimensional vector $u_\perp$ to employ a complex representation according to which $u_\perp = u + iv$, so that, for instance, $B \times u_\perp = iu_\perp$. Due to the fact that the related vectors are transforms, they in turn are already complex, and an evident confusion arises -- specifically, over the presence of the different nature of the two origins of i. The answer to this problem is that the procedure is consistent, ald all i's are such that $i^2=-1$. That this is so follows from writing out the full coordinate representation of the vector equations, and showing that the related $n \times n$ determinant may be written as $c\,\bar{c}$, where c in an $(n/2 \times n/2)$ determinant and $\bar{c}$ is its complex conjugate. The determinants which arise are of the form

$$D = \begin{vmatrix} a & b & c & m \\ -b & a & -m & c \\ p & e & f & g \\ -e & p & -g & f \end{vmatrix} = c\,\bar{c}\ ,$$

where c is the 2×2

$$c = \begin{vmatrix} a + ib & c + im \\ & \\ p + ie & f + ig \end{vmatrix}$$

The submatrices are the forms which arise in the complex representation. More generally, the statement is easily extended to D of dimension n (n even).

Formula (1) is seen to be a generalization of Sovillart's observation[37] that when $p = -e$, $e = m$, $f = a$, $g = -b$; then $D = (a^2 + b^2 + c^2 + d^2)^2$.

Appendix F: The Transfer Equations of a Plasma.*

Let the charge and mass of the first and second components of the plasma be $(\hat{e},\hat{m})$ and $(-e,m)$ respectively, where $e,\hat{e} > 0$. The two Boltzmann equations appear as

$$(1) \quad \frac{\partial \hat{f}}{\partial t} + \xi \cdot \nabla \hat{f} + \hat{e}/\hat{m}\ (E + \xi \times B) \cdot \nabla_{\xi} \hat{f} = \hat{j}(\hat{f},\hat{f}) + \hat{\text{ɉ}}(\hat{f},f) = \hat{\bar{j}}$$

$$(2) \quad \frac{\partial f}{\partial t} + \xi \cdot \nabla f - e/m\ (E + \xi \times B) \cdot \nabla_{\xi} f = j(f,f) + \text{ɉ}(f,\hat{f}) = \bar{j} \quad .$$

---

* This problem is considered in detail in reference (38).

The j terms represent the collision integrals.[31]

Consider the change in variables

$$(3) \qquad \begin{aligned} F &= \hat{m}\hat{f} + mf & \hat{f} &= \frac{eF + mG}{m\hat{e} + \hat{m}e} \\ G &= \hat{e}\hat{f} - ef & f &= \frac{\hat{e}F - \hat{m}G}{m\hat{e} + \hat{m}e} \ . \end{aligned}$$

The Boltzmann equations for F and G are

$$(4) \qquad \frac{\partial F}{\partial t} + \xi \cdot \nabla F + (E + \xi \times B) \cdot \nabla_{\xi} G = \hat{m}\hat{\bar{j}} + m\bar{j}$$

$$(5) \qquad \frac{\partial G}{\partial t} + \xi \cdot \nabla G + (E + \xi \times B) \cdot \nabla_{\xi} (\alpha F + \beta G) = \hat{e}\hat{\bar{j}} - e\bar{j} \ .$$

The constants α and β are given by

$$(6) \qquad \alpha = \hat{e}e/\hat{m}m \ , \qquad \beta = \frac{e^2\hat{m}^2 - e^2\hat{m}^2}{m\hat{m}(e\hat{m} + \hat{e}m)} \ ,$$

so that

$$(7) \qquad \begin{aligned} \alpha(\hat{e},\hat{m};\ e,m) &= \alpha(e,m;\ \hat{e},\hat{m}) \\ \beta(\hat{e},\hat{m};\ e,m) &= -\beta(e,m;\ \hat{e},\hat{m}) \ . \end{aligned}$$

If $e = \hat{e}$ and $m = \hat{m}$, β vanishes.

In the following, $\langle A \rangle$ represents the integral of A over velocity space. The mean mass flow U is defined according to

(8) $$\rho u = \langle F\xi \rangle ,$$

where ρ is the mass density,

(9) $$\rho = \langle F \rangle .$$

The extraordinary velocity c is given by,

(10) $$c = \xi - u .$$

The charge density Q is

(11) $$Q = \langle G \rangle .$$

The conduction current J is related to the total current J' through

(12) $$J = \langle Gc \rangle = \langle G(\xi - u) \rangle = J' - uQ .$$

The pressure P_{ij} and "J-flux" R_{ij} tensors are given by

(13) $$P_{ij} = \langle F\ c_i c_j \rangle , \qquad R_{ij} = \langle G\ c_i c_j \rangle .$$

The internal energy W and its charge counterpart $\tilde{R}$, are given by

(14) $$W = \frac{1}{2} \langle Fc^2 \rangle = \frac{1}{2} \mathrm{Tr}\ P_{ij} , \qquad \tilde{R} = \frac{1}{2} \langle Gc^2 \rangle = \frac{1}{2} \mathrm{Tr}\ R_{ij} .$$

Finally, the heat flux Z and the $\tilde{R}$ flux K are given by

(15) $$Z = \frac{1}{2} \langle Fc^2 c \rangle , \qquad K = \frac{1}{2} \langle Gc^2 c \rangle .$$

The equations governing these variables are obtained by operating on equations (4) and (5) with $\int d^3\xi$, $\int d^3\xi c$, $\frac{1}{2}\int d^3\xi c^2$. (In forming these equations we have recalled that $\langle Fc\rangle = \frac{\partial\xi}{\partial x} = \frac{\partial\xi}{\partial t} = 0$, $\frac{\partial c_i}{\partial\xi_j} = \frac{\partial\xi_i}{\partial\xi_j} = \delta_{ij}$.)

The continuity equations:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho u = 0 \tag{16}$$

$$\frac{\partial Q}{\partial t} + \nabla\cdot J = 0 \quad .$$

The momentum and current equations:

$$\rho\left(\frac{\partial}{\partial t} + u\cdot\nabla\right)u + \nabla\cdot P_{ij} - Q(E + u\times B) - J\times B = 0$$

$$\left(\frac{\partial}{\partial t} + \nabla\cdot u\right)J + \left\{Q\,\frac{\partial}{\partial t} + (J + uQ)\cdot\nabla\right\}u + \nabla\cdot R_{ij} \tag{17}$$

$$- \rho\alpha(E + u\times B) - \beta(EQ + J\times B) = \left\langle c(\hat{e}\hat{\bar{j}} - e\bar{j})\right\rangle \quad .$$

The energy and $\tilde{R}$ equations:

$$\left(\frac{\partial}{\partial t} + \nabla\cdot u\right)W + \nabla\cdot Z + P : \nabla u - J\cdot[E + u\times B] = 0$$

$$\left(\frac{\partial}{\partial t} + \nabla\cdot u\right)\tilde{R} + \nabla\cdot K + R : \nabla u - J\cdot\left[\left(\frac{\partial}{\partial t} - u\cdot\nabla\right)u \right. \tag{18}$$

$$\left. + \beta(E + u\times B)\right] = \frac{1}{2}\left\langle c^2(\hat{e}\hat{\bar{j}} - e\bar{j})\right\rangle$$

$$R : \nabla u = R_{ij}\nabla_i u_j \quad .$$

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